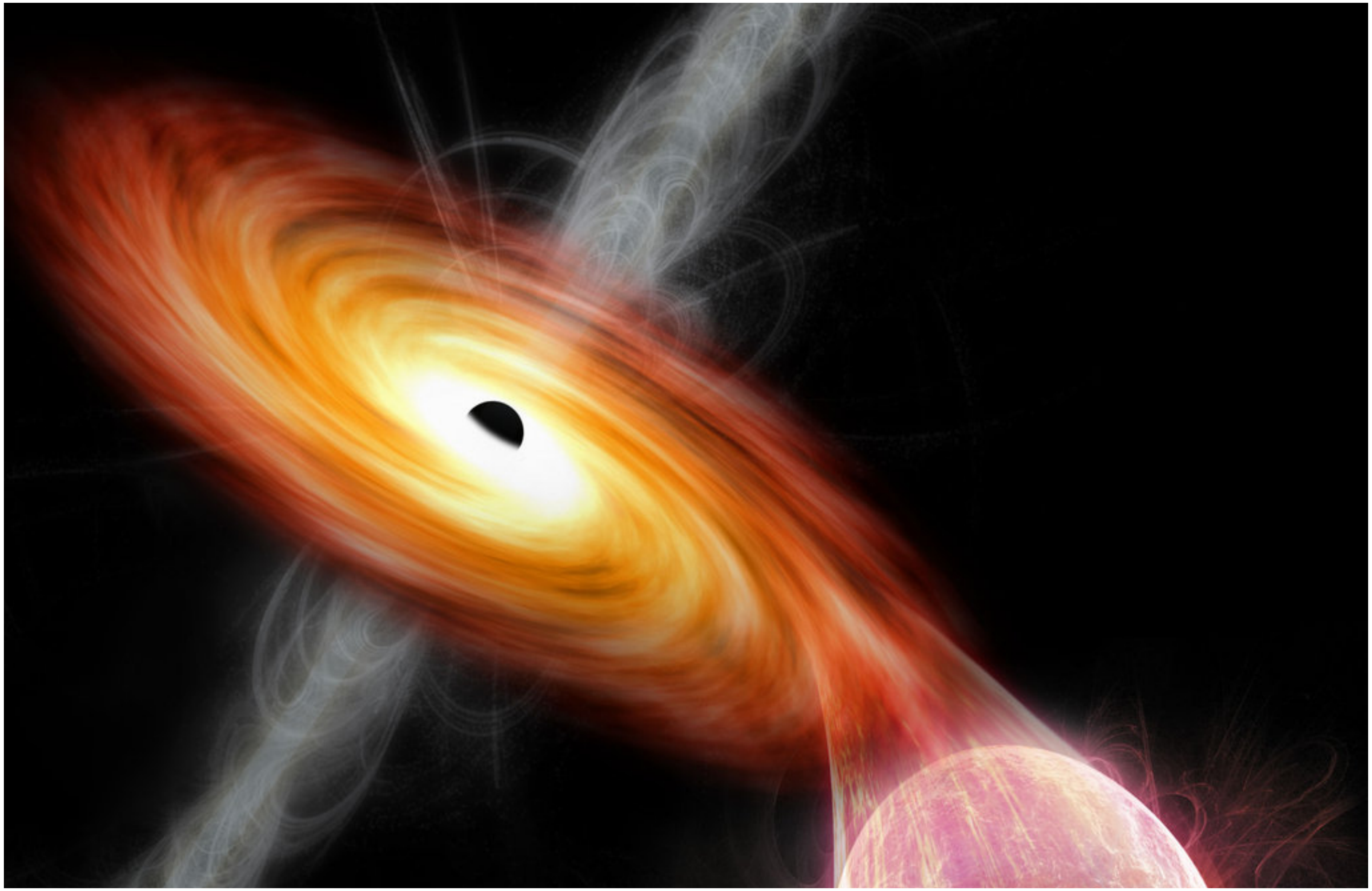
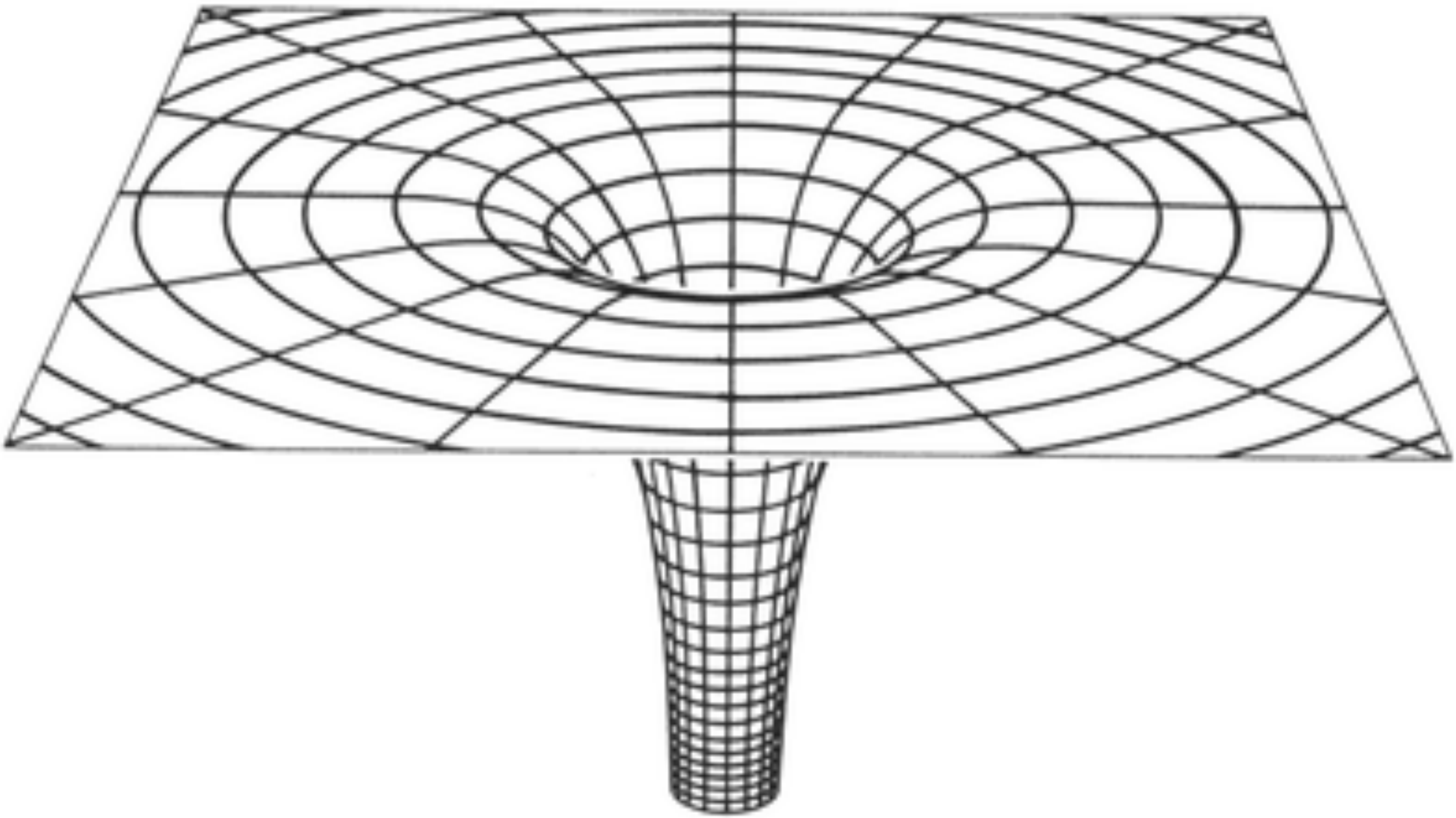
High Energy Astrophysics

Dr. Adam Ingram

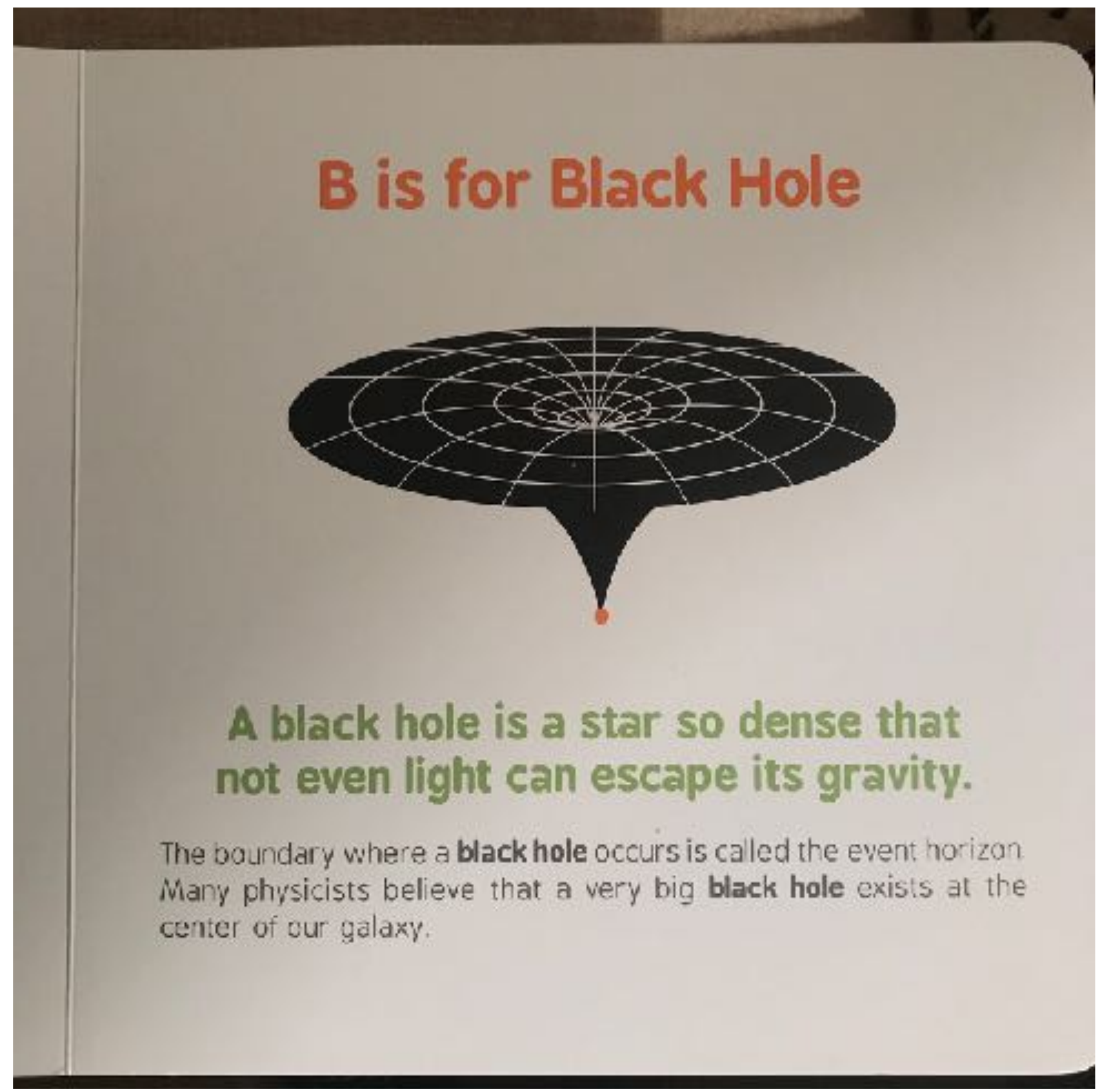


Lecture 5

Black Holes

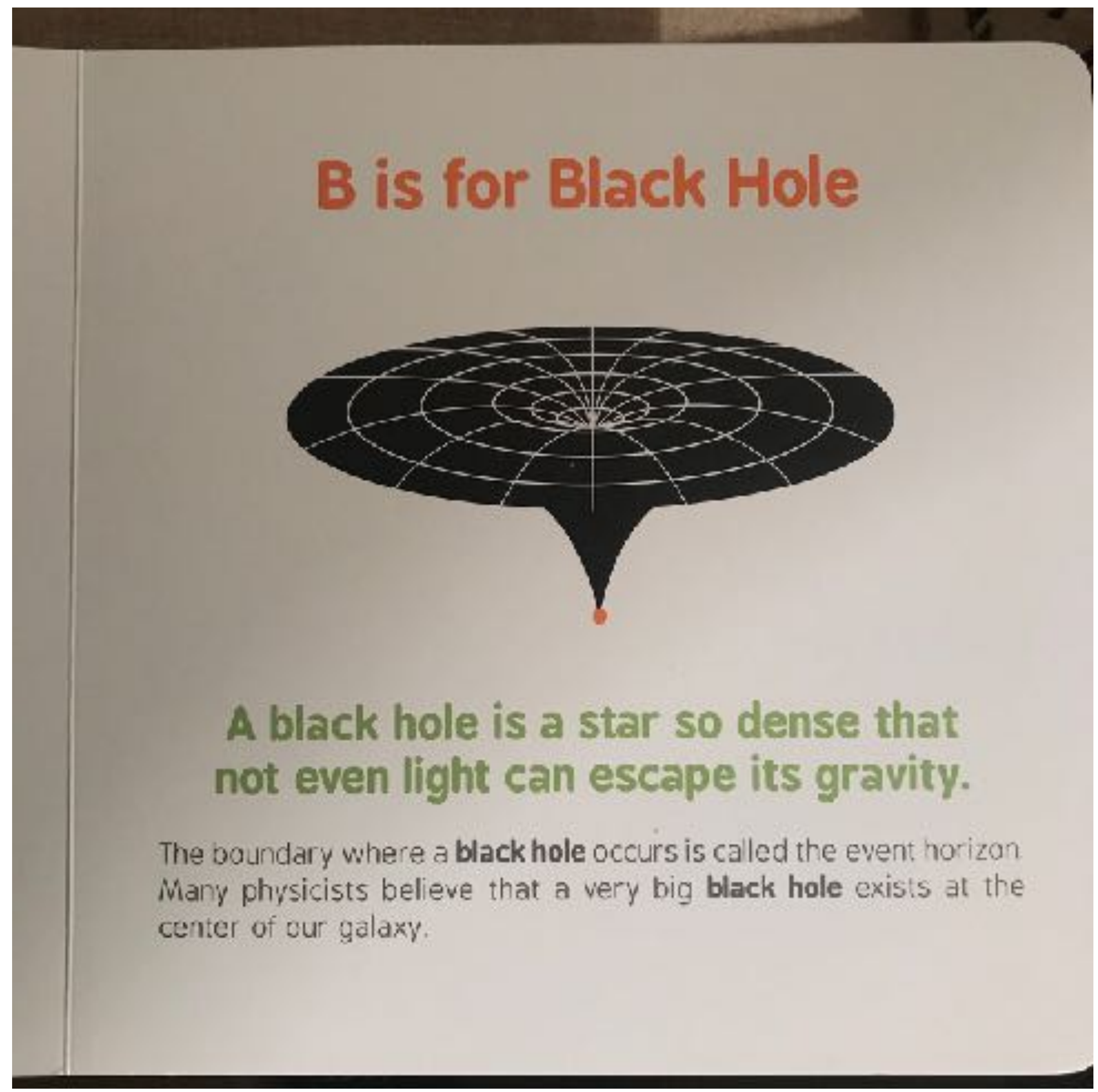


Black Holes for Babies



Black Holes for Babies

- All mass, M, in a singularity
- Event Horizon: $v_{\rm esc} = c$!
- Newtonian approx: $v_{\rm esc}^2 = 2GM/r \implies r_h = 2GM/c^2$
- Size scale: gravitational radius: $r_g = GM/c^2$



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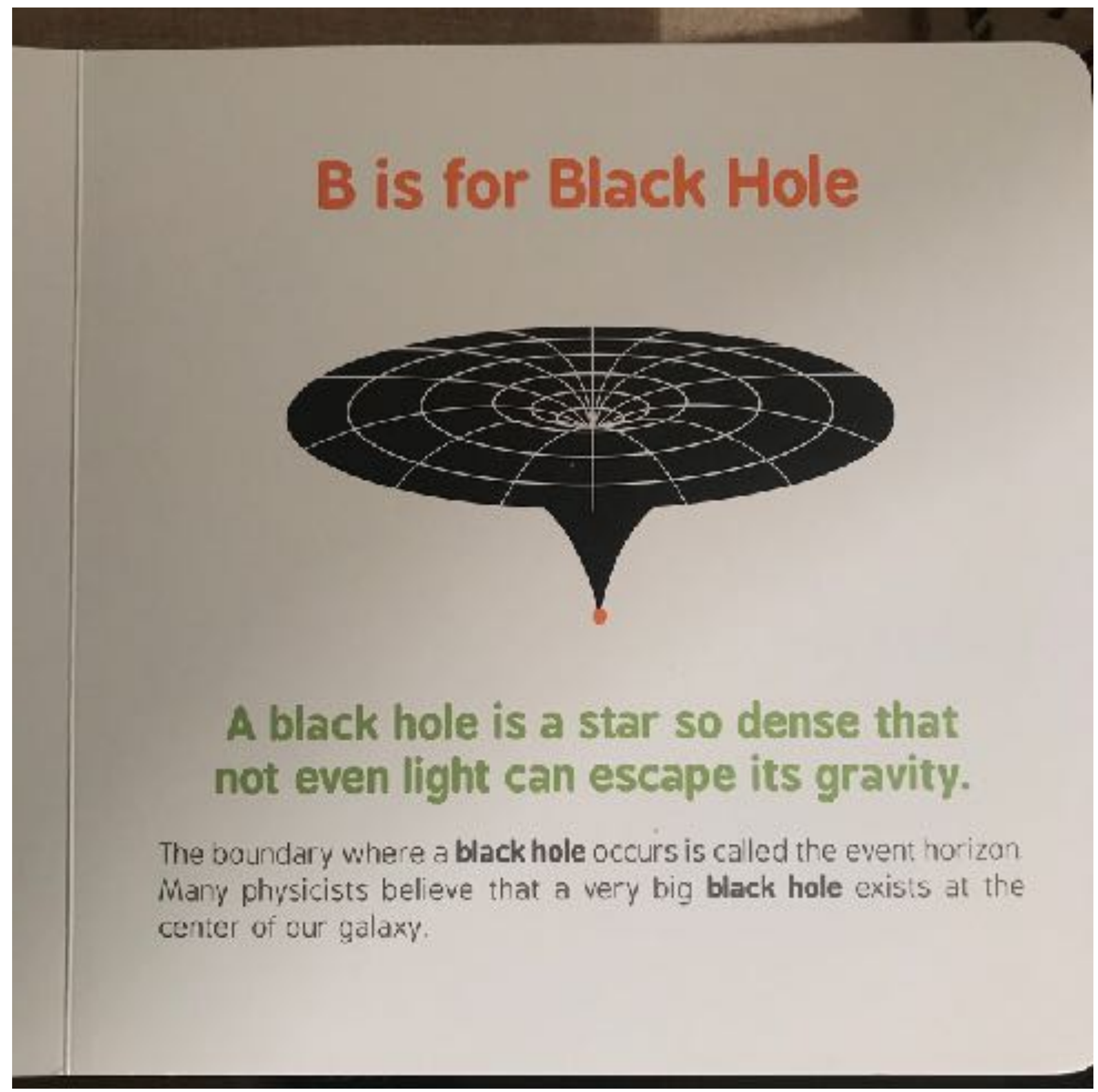
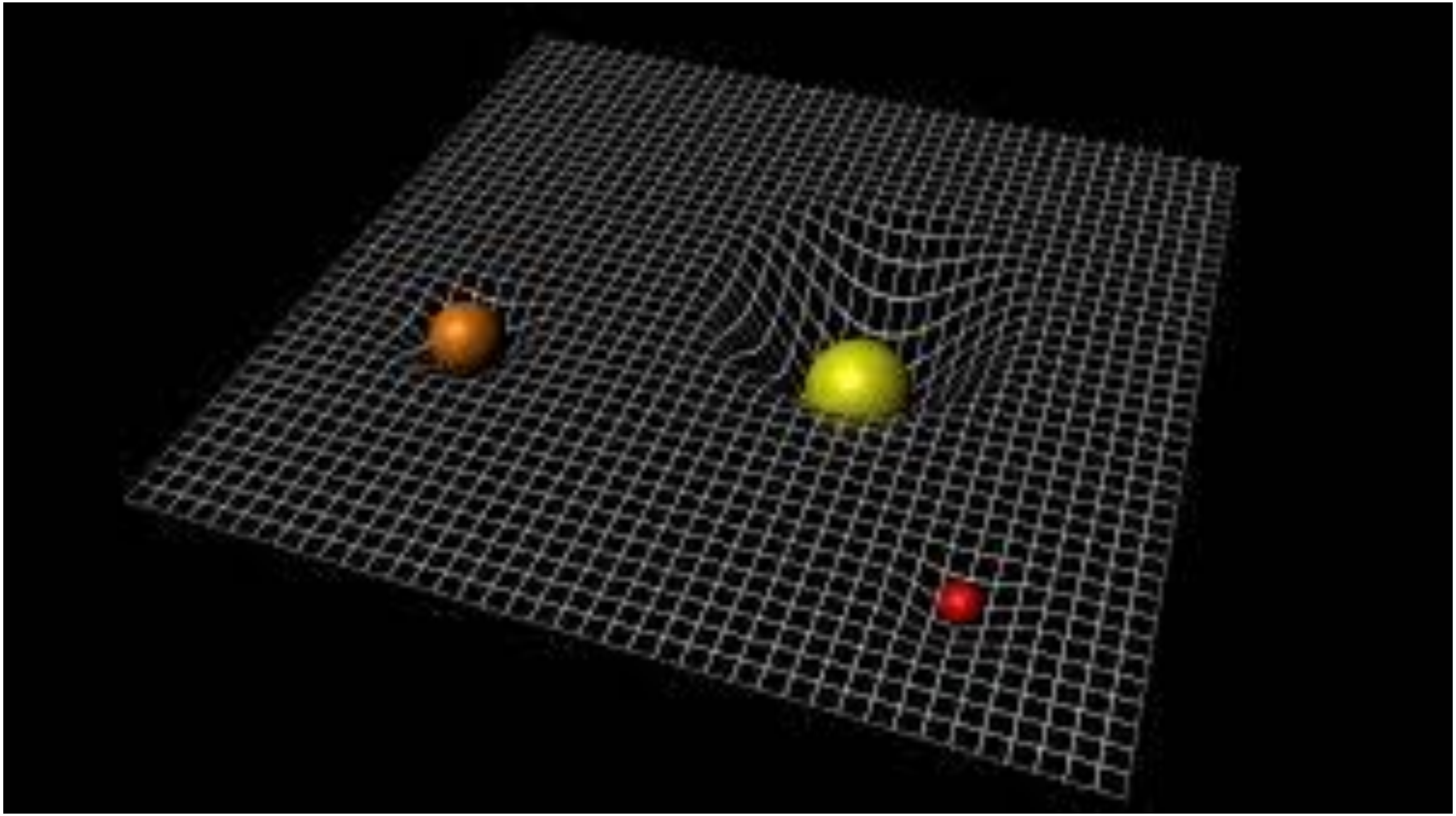
Symbol r_g was gyroradius; is now gravitational radius!



Black Holes in GR

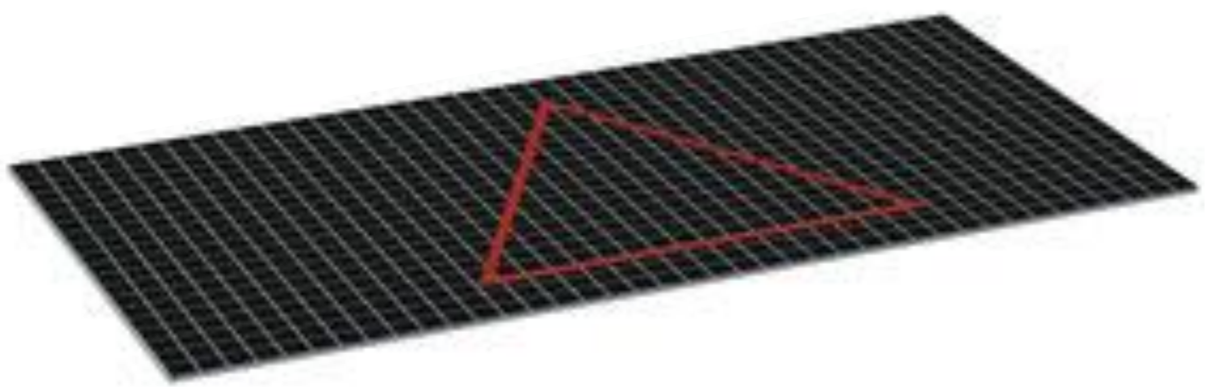
- In GR, gravity = curvature. That is, mass density induces curvature in spacetime, which affects trajectories of test masses.

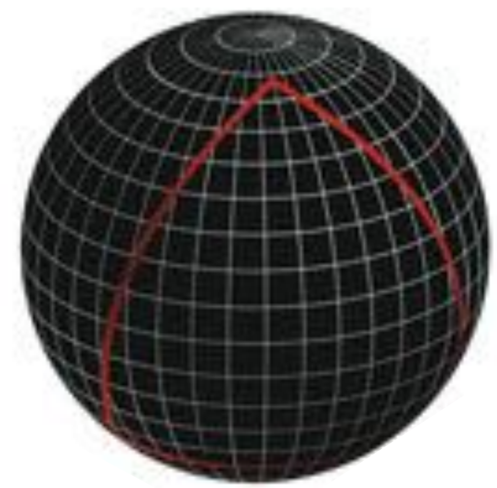


Black Holes in GR

- In GR, gravity = curvature. That is, mass density induces curvature in spacetime, which affects trajectories of test masses.
- What does spacetime curvature mean? Well, in flat 3D space we have that the distance between two points is given by Pythagorus' theorem:

$$(d\ell)^2 = (dx)^2 + (dy)^2 + (dz)^2 = (dr)^2 + (rd\theta)^2 + (r\sin\theta d\phi)^2$$





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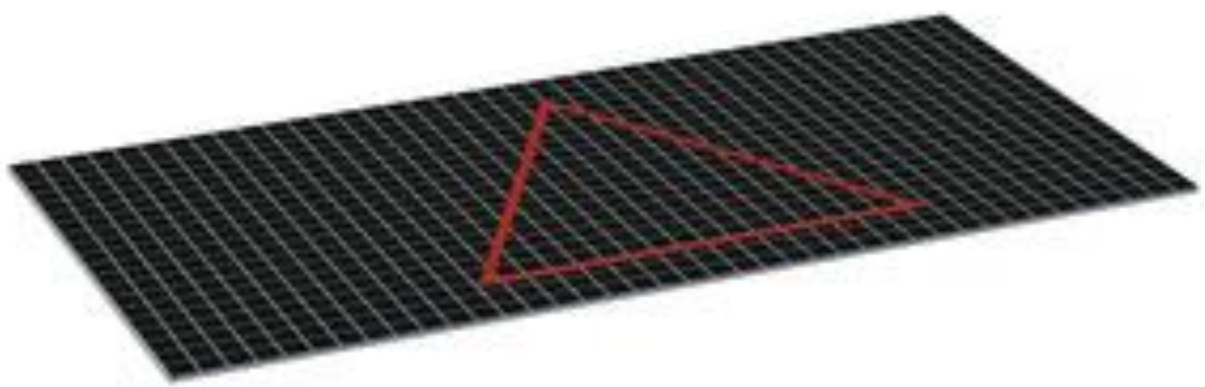

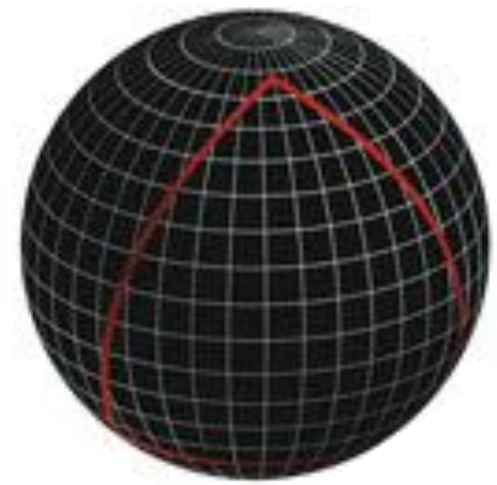

- In **curved** 3D space, it is not given by Pythagorus.
- In SR, include time into 4D spacetime — introduce **spacetime interval**. For flat spacetime, this is:

$$(ds)^2 = -(cdt)^2 + (dx)^2 + (dy)^2 + (dz)^2 = -(cdt)^2 + (dr)^2 + (rd\theta)^2 + (r\sin\theta d\phi)^2$$

- Position of minus sign is just a choice.

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- Can either have minus sign in front of coordinate time and proper time or in front of spatial coordinates.

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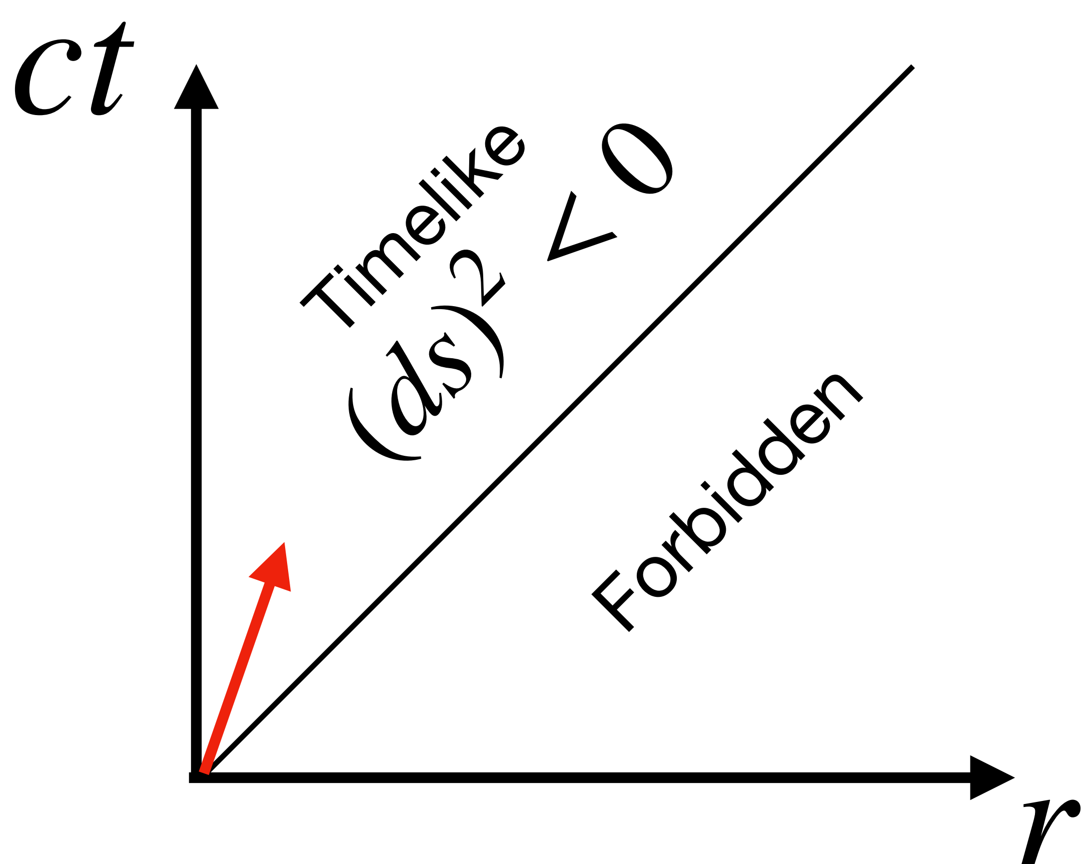
- Allowed worldlines are **timelike** — I.e. involve travelling slower than c.
- Timelike = separated by more time than space: $(ds)^2 < 0$



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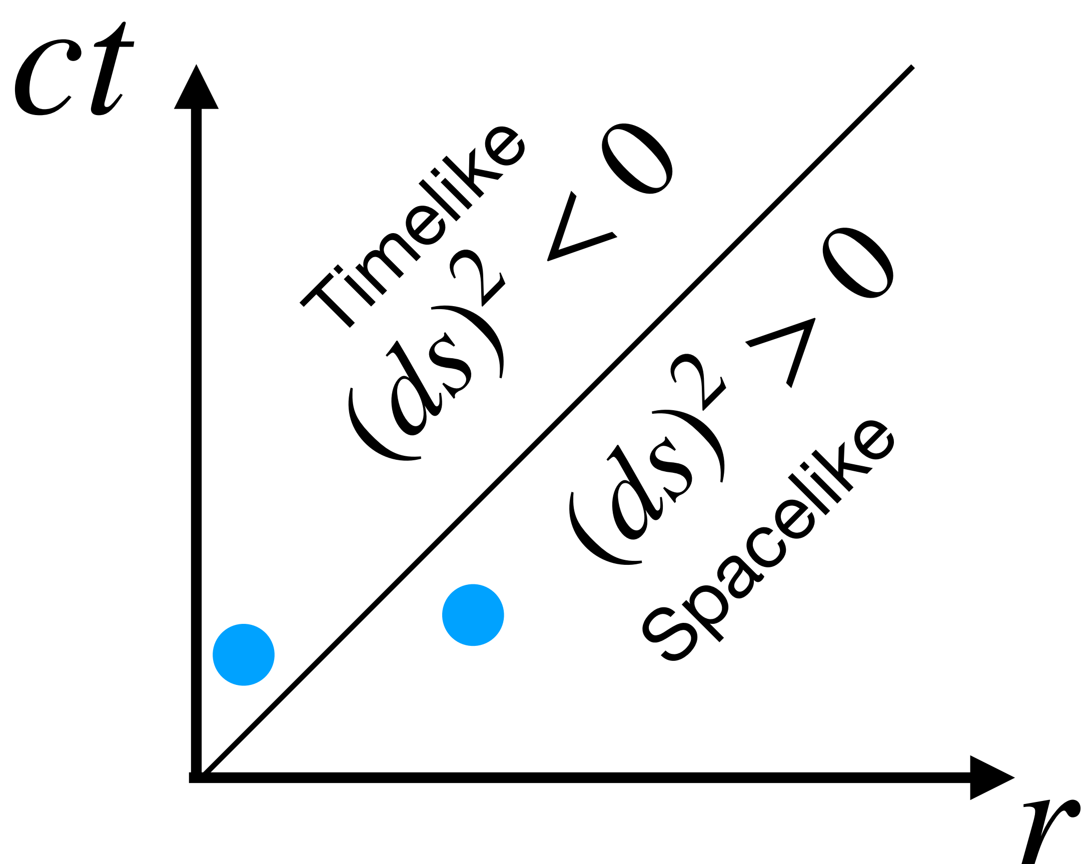

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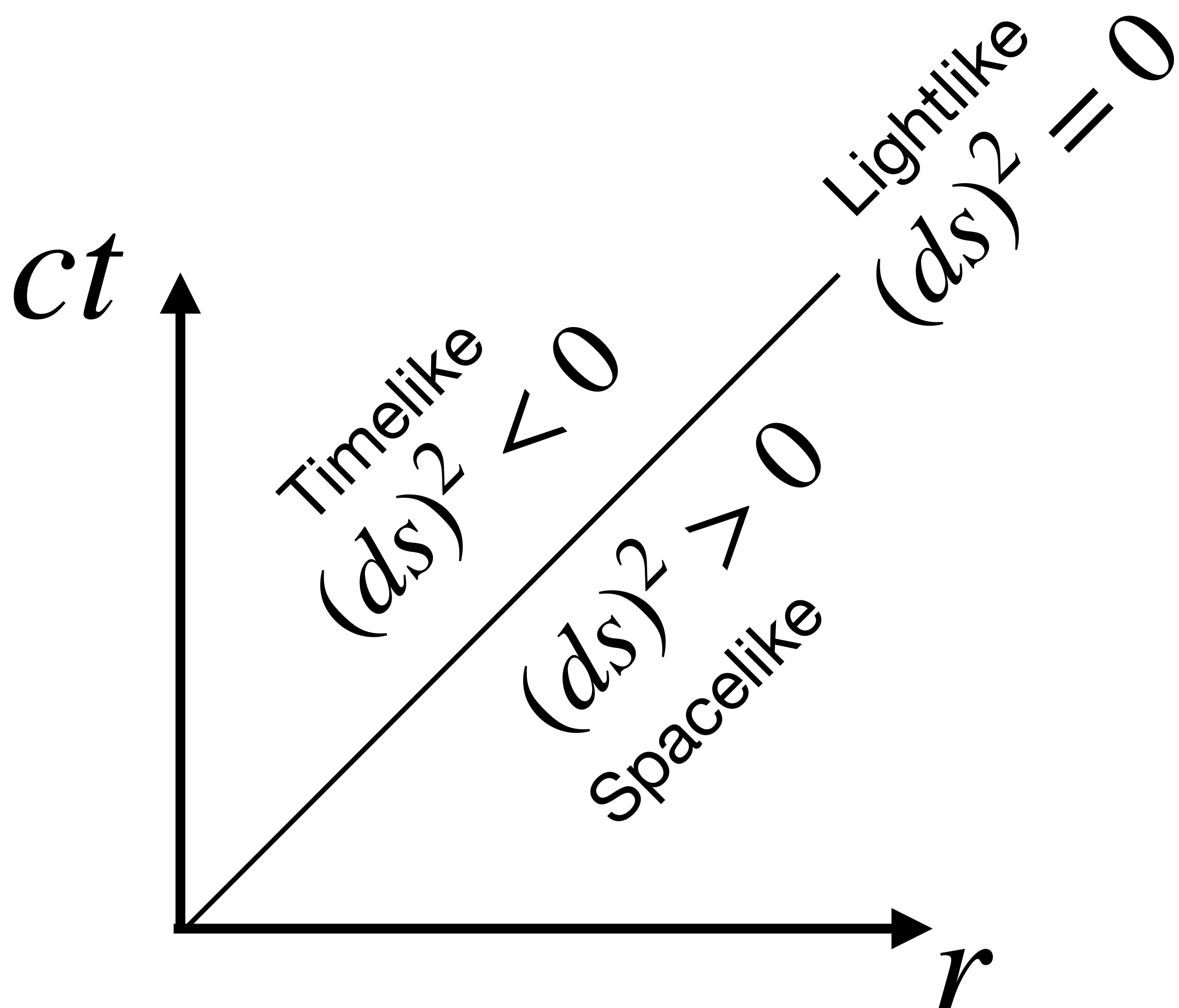
- Light travels at c on null worldlines: $(ds)^2 = 0$



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$A^\mu = \text{vector}$ $A^{\mu\nu} = \text{2D matrix}$ $A^{\mu\nu\sigma} = \text{3D matrix}$

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- Therefore spacetime interval (=line element):

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- Cartesian:

$$\eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

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- Metric depends on spacetime curvature, but also on coordinate system. Line element (spacetime interval) is independent of coordinate system.

The Einstein Field Equations

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Einstein tensor
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Stress-energy tensor
= mass density and pressure
In SR, this is:

$$T_{\mu\nu} = \begin{pmatrix} \rho_0 c^2 & 0 & 0 & 0 \\ 0 & P_x & 0 & 0 \\ 0 & 0 & P_y & 0 \\ 0 & 0 & 0 & P_z \end{pmatrix}$$

The Schwarzschild Metric

$$G_{\mu\nu} = 8\pi \frac{G}{c^4} T_{\mu\nu}$$

- Solve Einstein equations for spherically symmetric spacetime with $\rho(\mathbf{r}) = M\ \delta^3(\mathbf{r})$
- i.e. all mass concentrated at the singularity (a BH).

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- Line element becomes: $(ds)^2 = -\left(1 - \frac{2r_g}{r}\right)(cdt)^2 + \frac{(dr)^2}{1-2r_g/r} + (rd\theta)^2 + (r\sin\theta d\phi)^2$

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- Tends to Minkowski for large r, but for small r angles of a triangle no longer add up to 180 degrees!

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- r=0 singularity is real, r=$2r_g$ one is only a coordinate singularity (goes away with clever choice of coordinate system).

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- The same as our naive Newtonian guess!

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- Now let's do orbits in the Schwarzschild metric.
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- Circular orbit: dr/dt=0, so V(r)=E.
- E=constant, so for a circular orbit dV(r)/dr=0: circular orbits at turning points of V(r).
- Minima = stable, maxima and inflection points = unstable.

The Innermost Stable Circular Orbit

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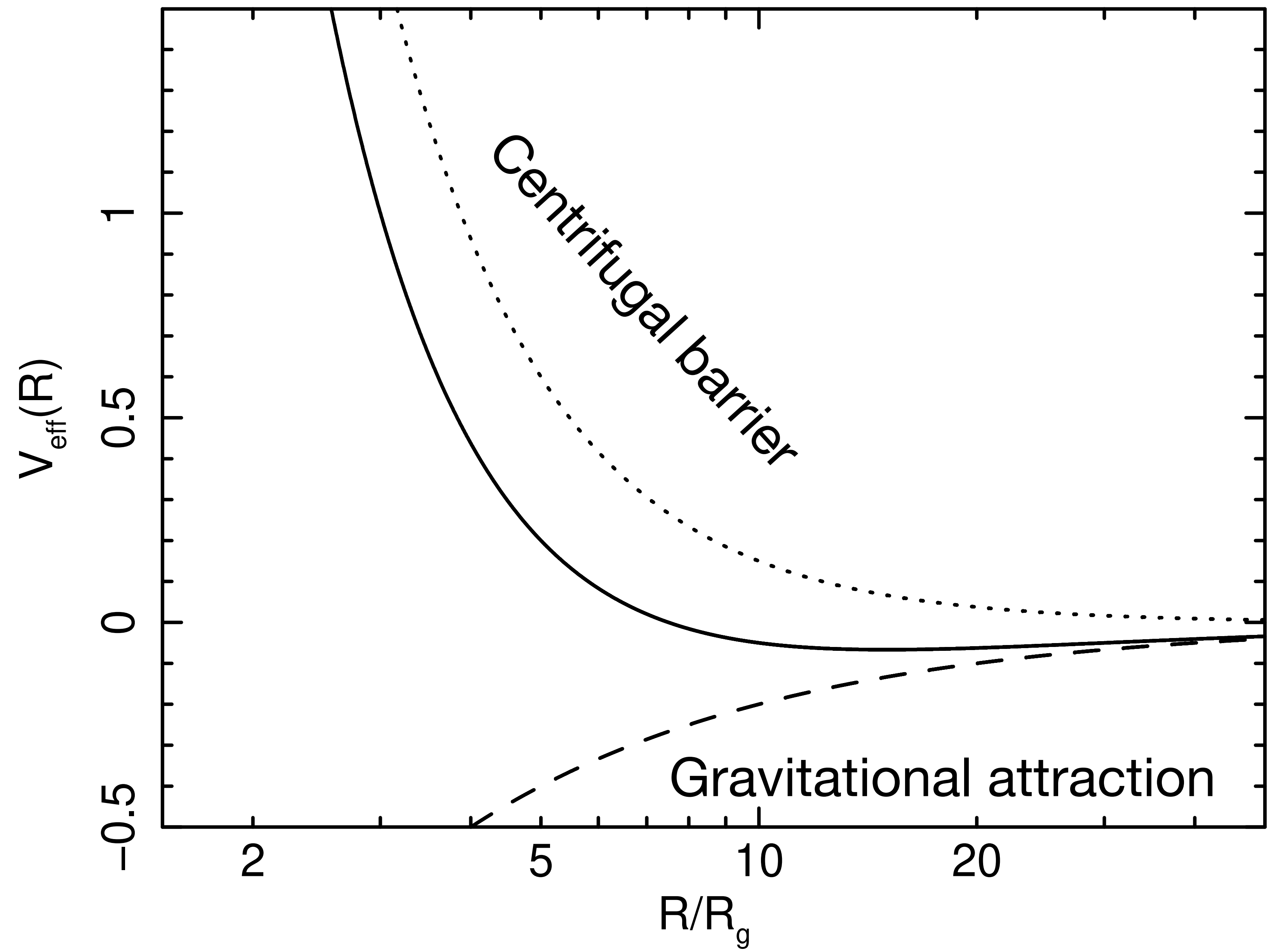

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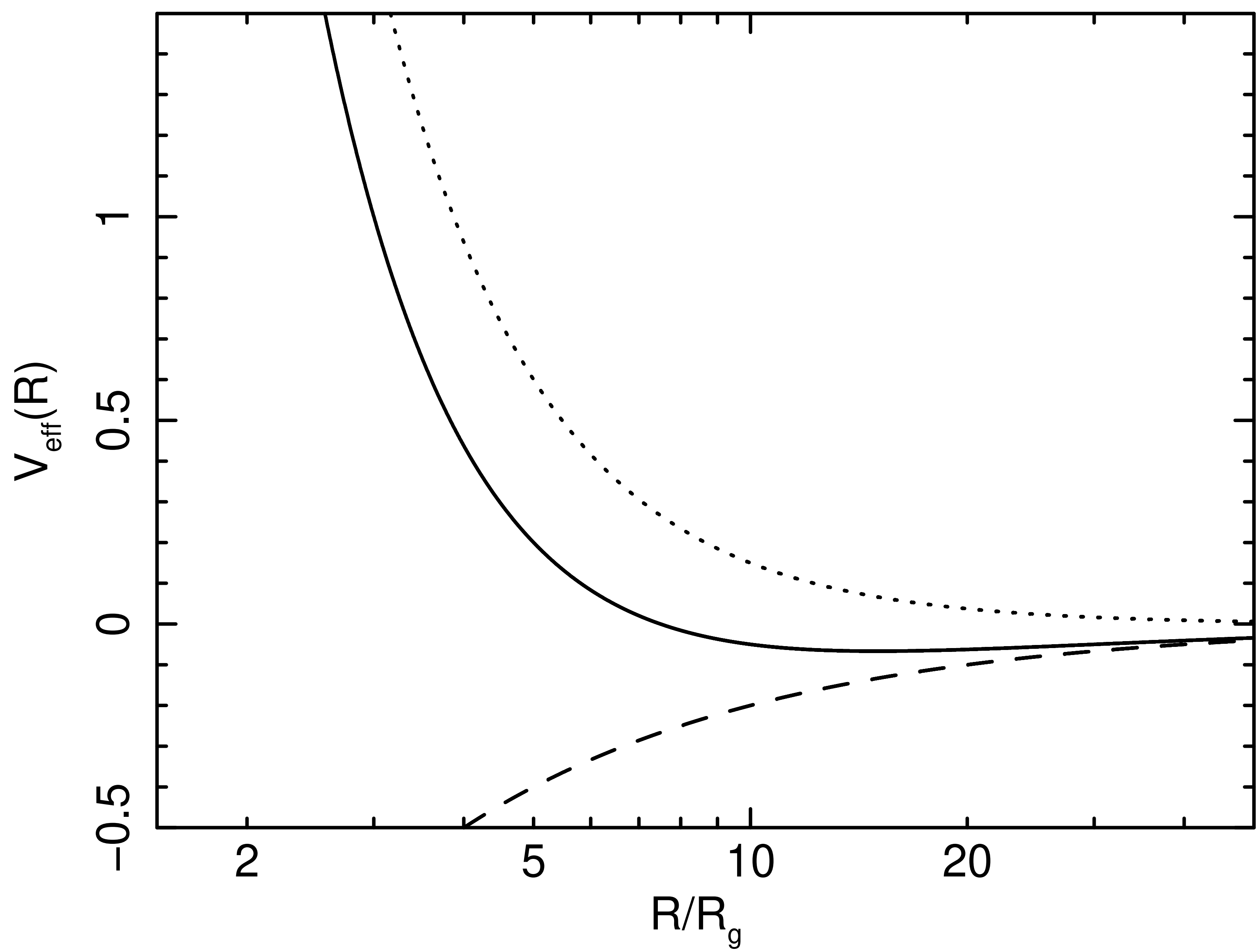

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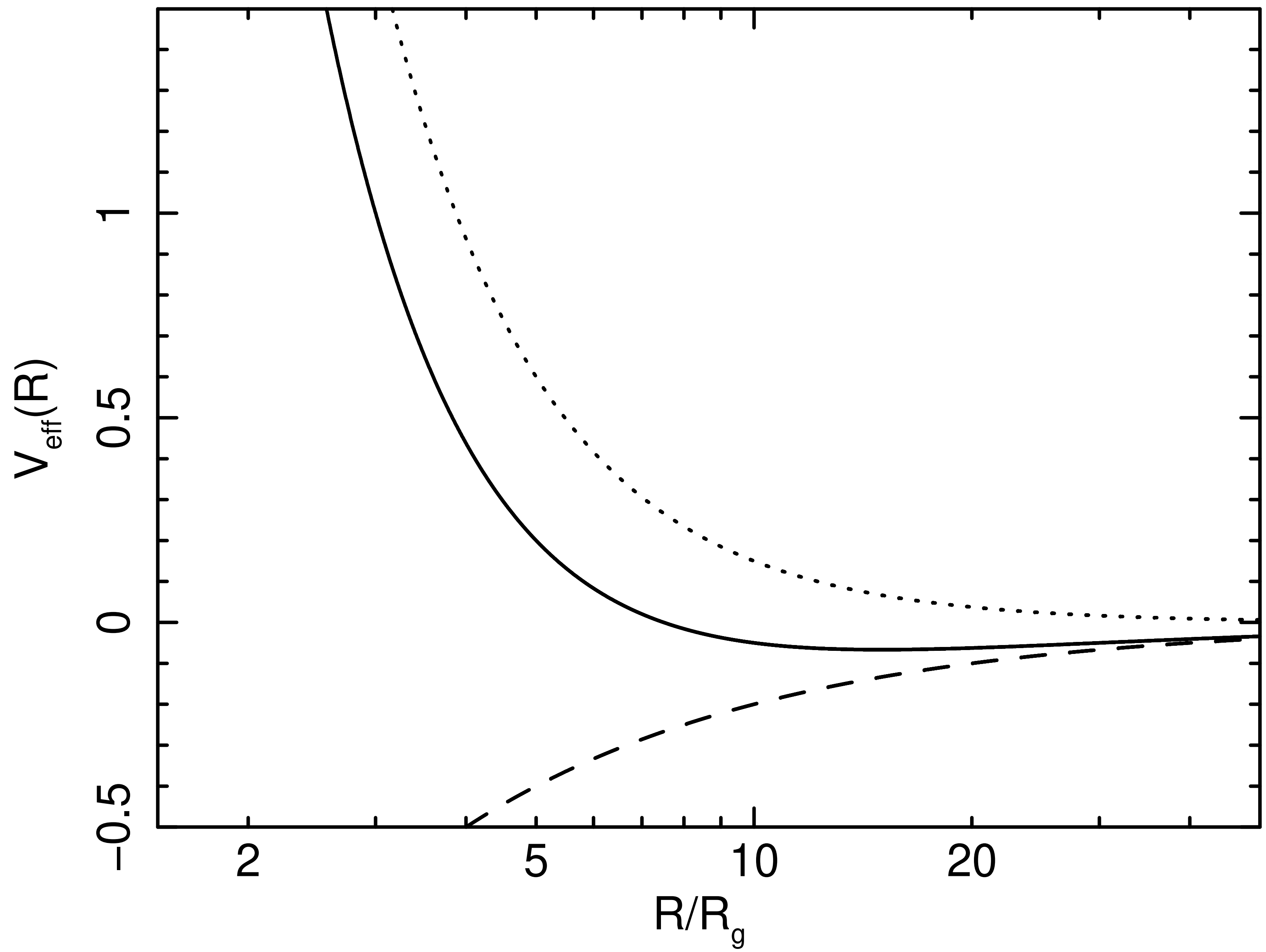

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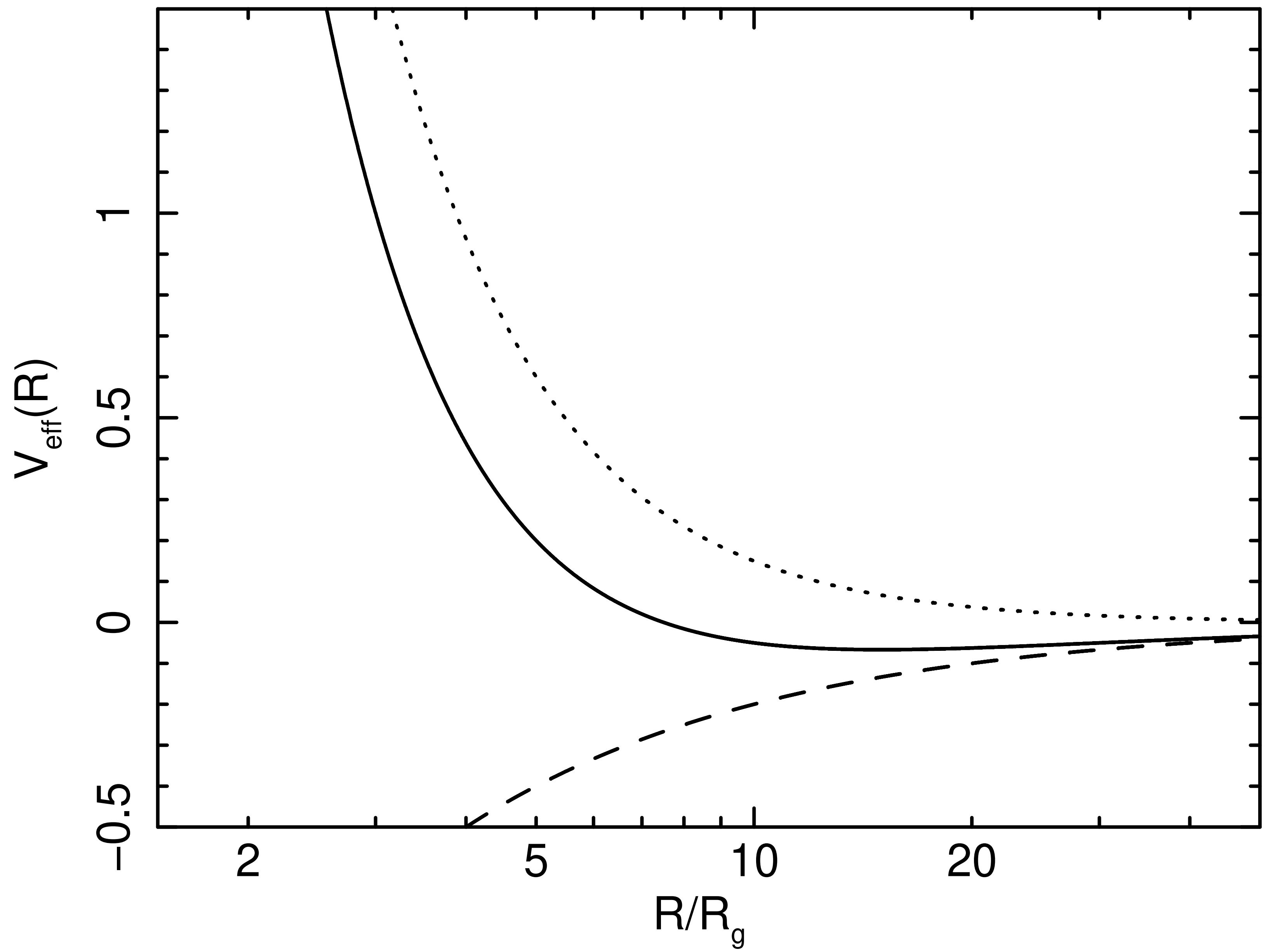

$$L = vr \quad \therefore v^2r^2 = GMr \quad \therefore v^2 = GM/r$$

…Keplerian orbit!

The Innermost Stable Circular Orbit

- For Schwarzschild solution, energy equation becomes:

$$\left(\frac{dr}{d\tau}\right)^2 + V_{\rm eff}^2(r) = \left(\frac{E}{c}\right)^2$$

- With effective potential:

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Centrifugal barrier

Gravitational attraction

Completely new!

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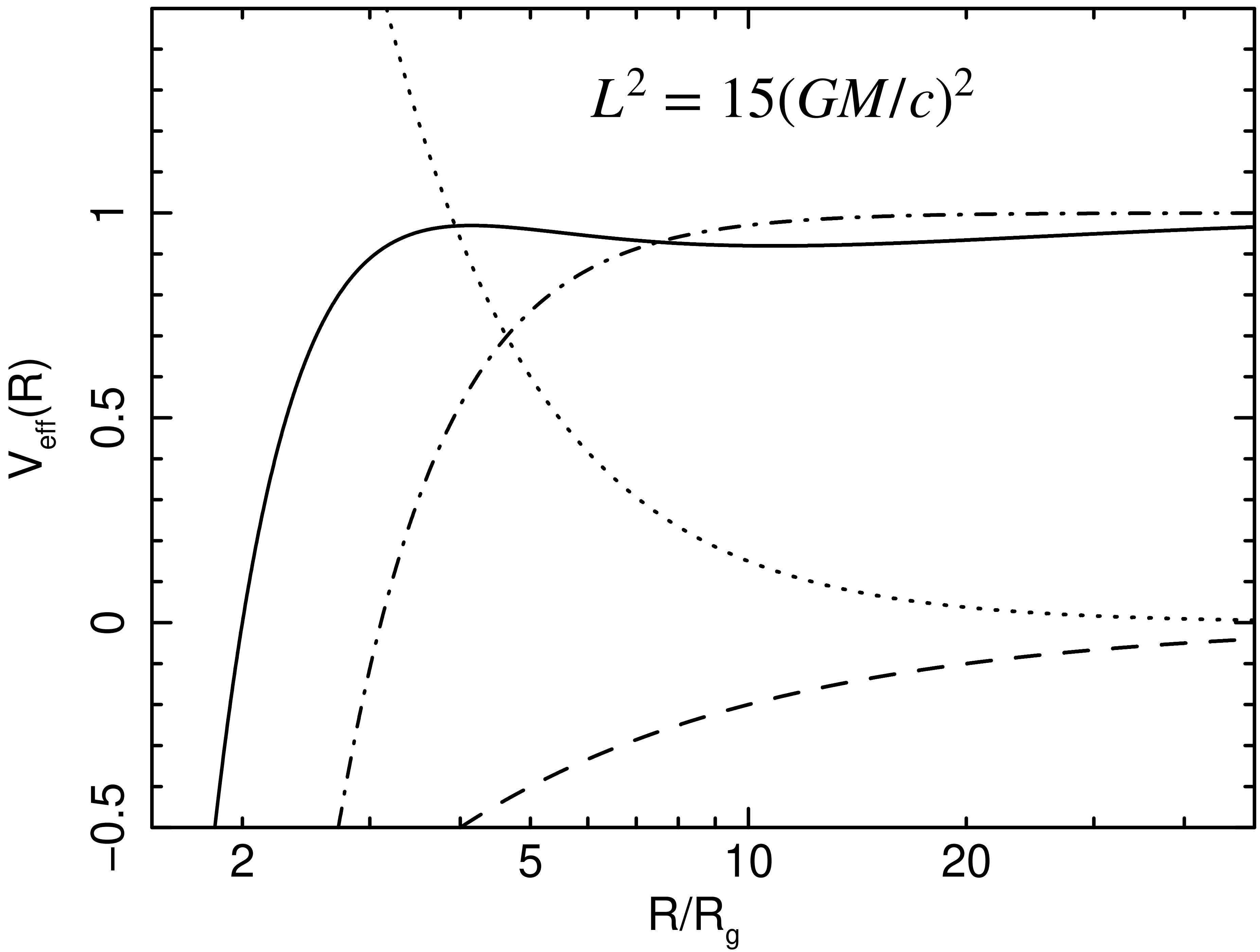

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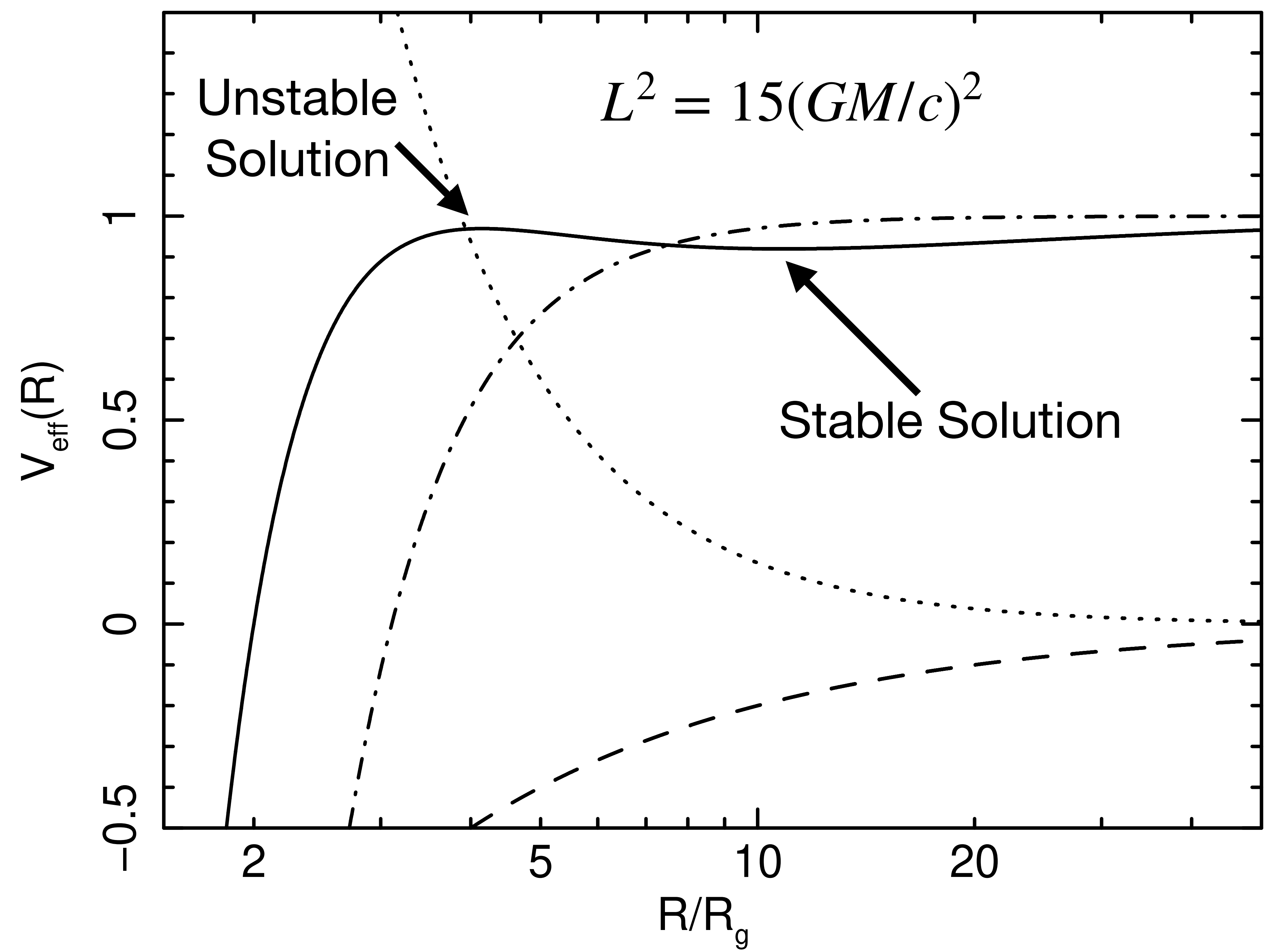

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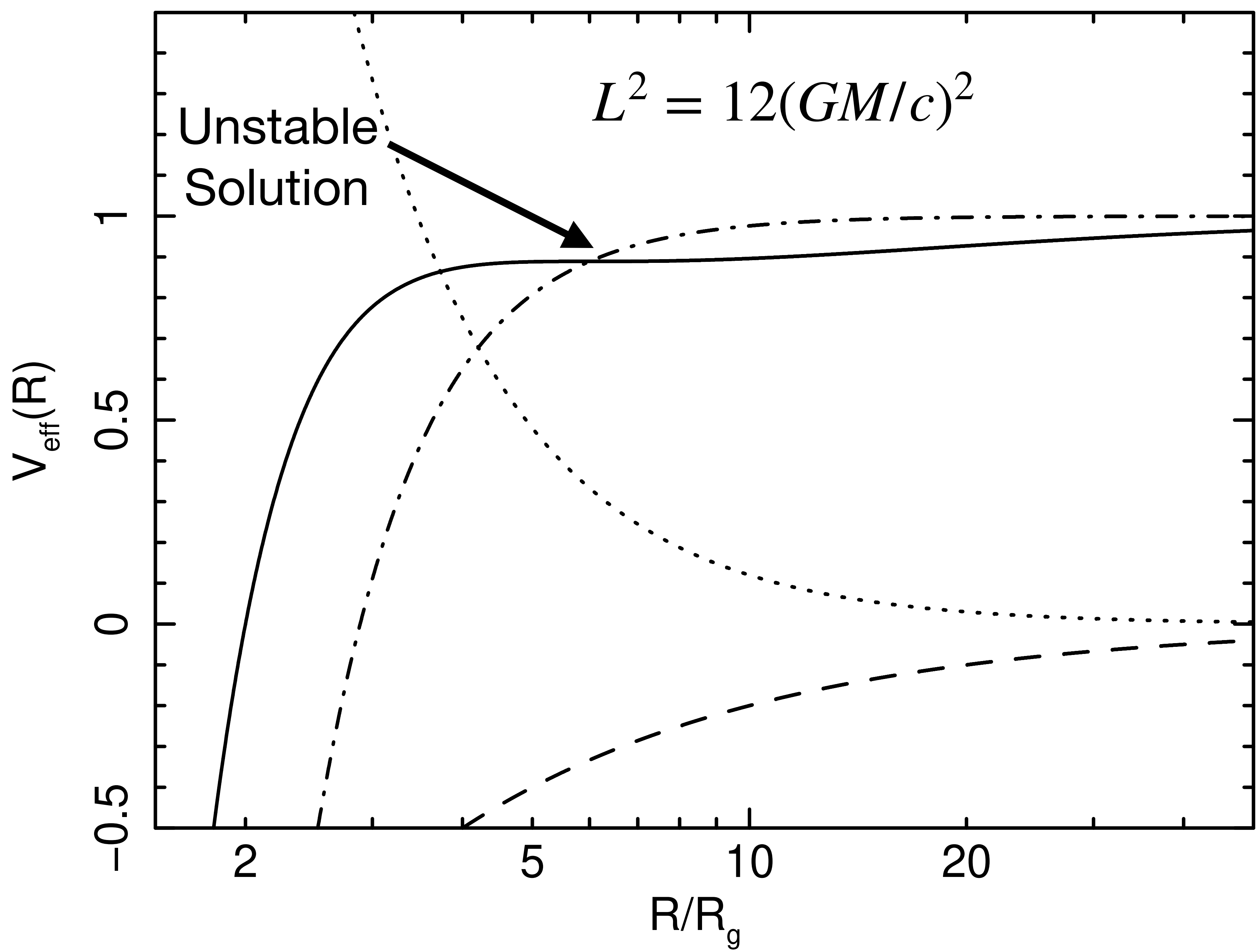

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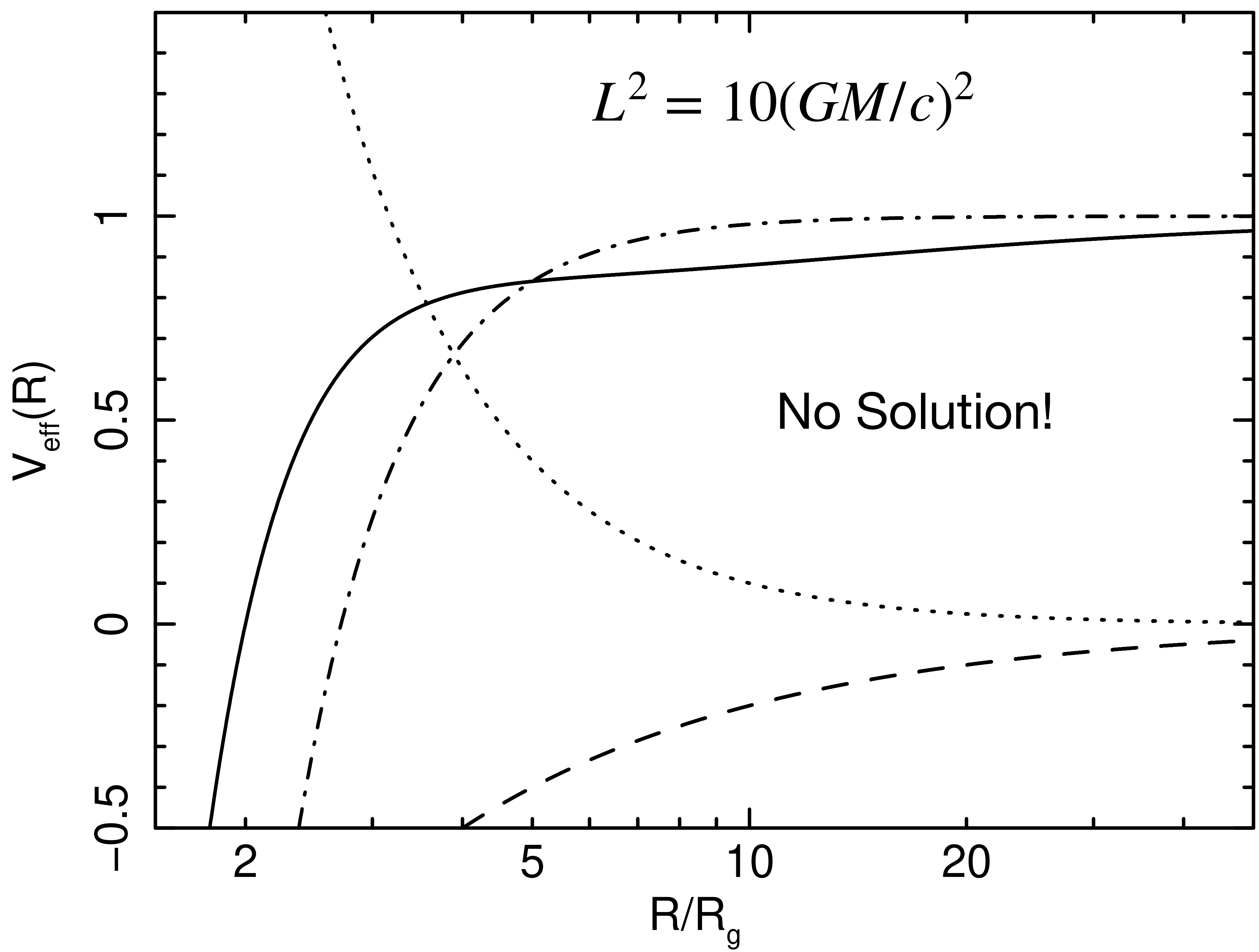

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…two solutions: stable and unstable!

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- No solutions for: $L < L^2_{\rm crit}$
- Therefore ISCO at:

$$r_{\rm isco} = \frac{L^2_{\rm crit}}{2GM} = \frac{12(GM)^2}{2GMc^2} = 6\ r_g$$

- Inside 6 r_g, can still in principle escape the BH, but can’t orbit without help!

The Kerr Metric

- If the BH is spinning, no longer have spherical symmetry, only azimuthal.
- Kerr solution (Kerr 1960) is:

$$g_{tt} = -\left(1 - \frac{2r}{r_g \Sigma}\right) \quad g_{t\phi} = g_{\phi t} = \frac{2ar\sin^2\theta}{r_g \Sigma} \quad g_{rr} = \frac{\Sigma}{\Delta} \quad g_{\theta\theta} = \Sigma \quad g_{\phi\phi} = \frac{\mathscr{A}\sin^2\theta}{\Sigma}$$

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- Boyer-Lindquist coordinates:

$$x = r_g\sqrt{(r/r_g)^2 + a^2}\,\sin\theta\cos\phi$$

$$y = r_g\sqrt{(r/r_g)^2 + a^2}\,\sin\theta\sin\phi$$

$$z = r\cos\theta$$

The Kerr Metric

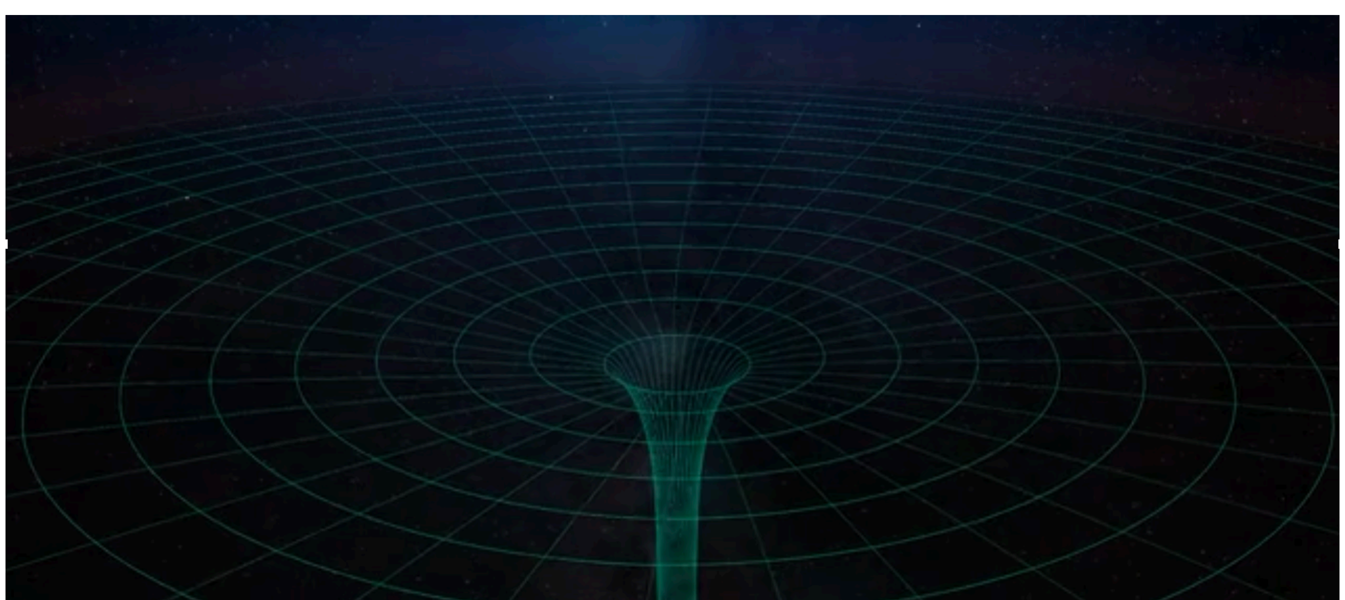
- Cross term = **Frame Dragging Effect**

$$g_{t\phi} = g_{\phi t} = \frac{2ar\sin^2\theta}{r_g\Sigma}$$

- Kerr BH drags spacetime around with it.

- Gives rise to (among other things) Lense-Thirring precession — a vertical wobble of orbits in a plane inclined to the BH equatorial plane.

a = 0



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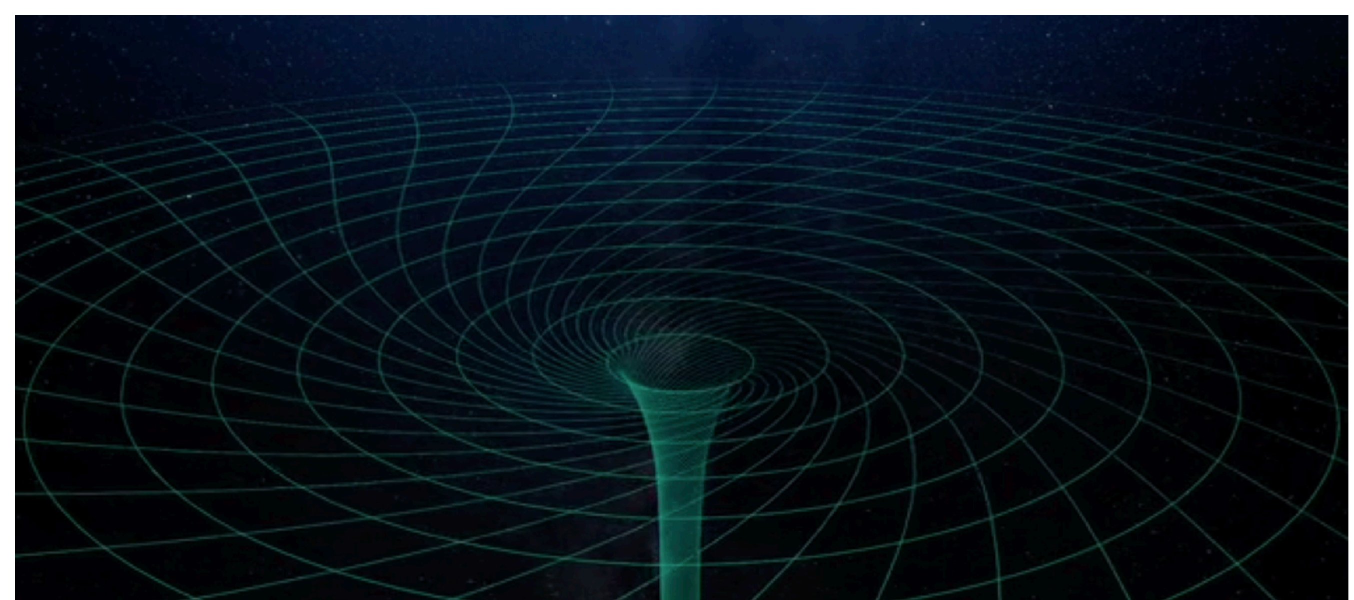

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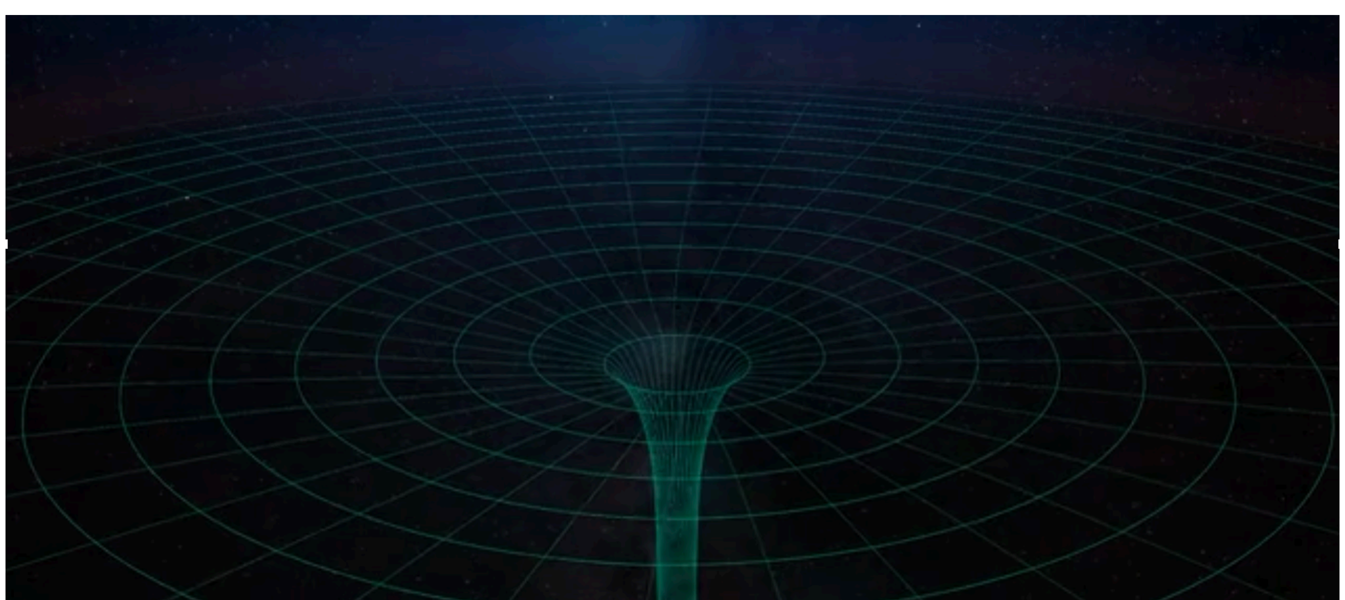

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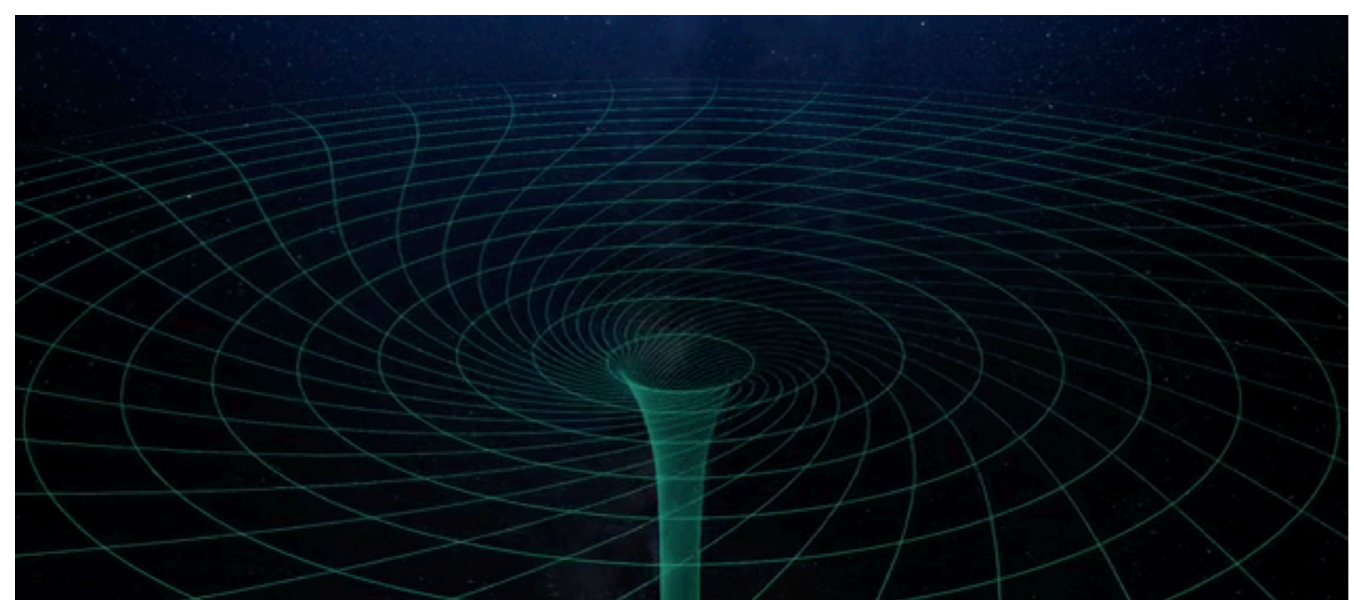

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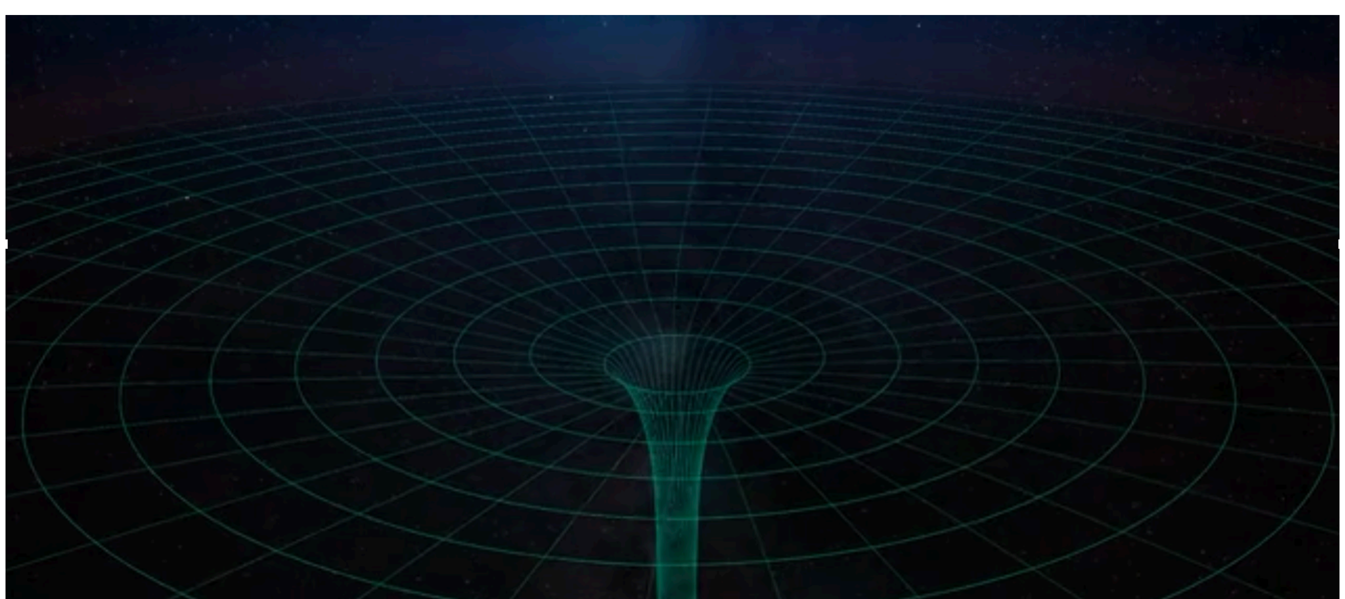

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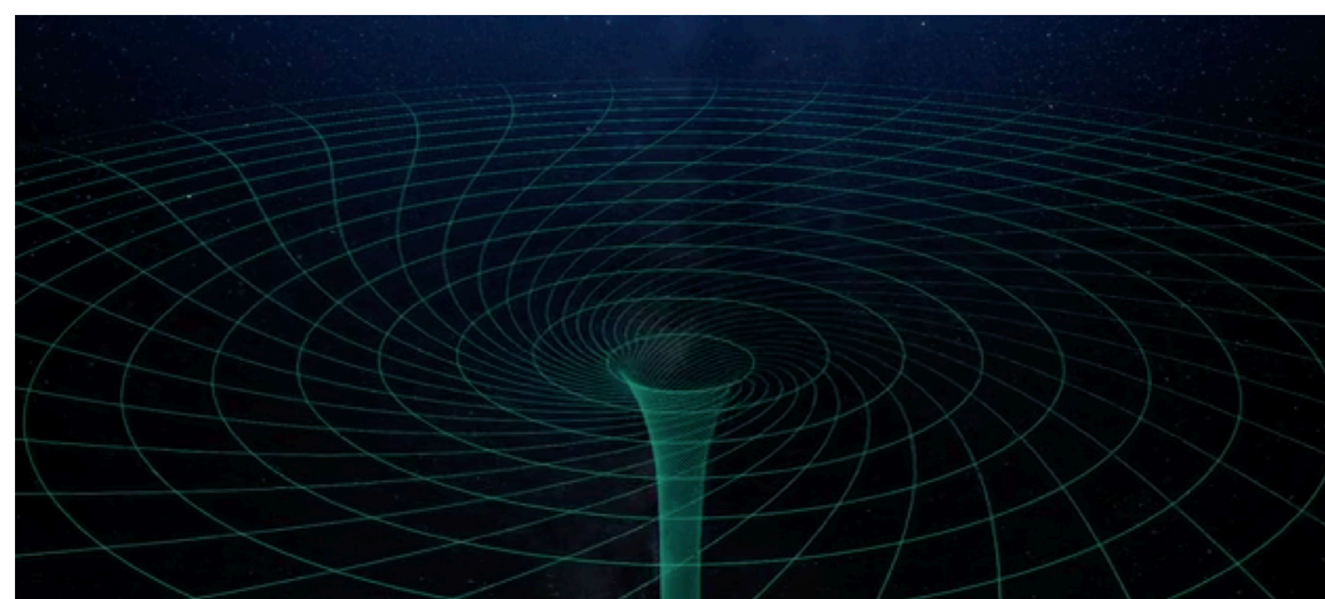

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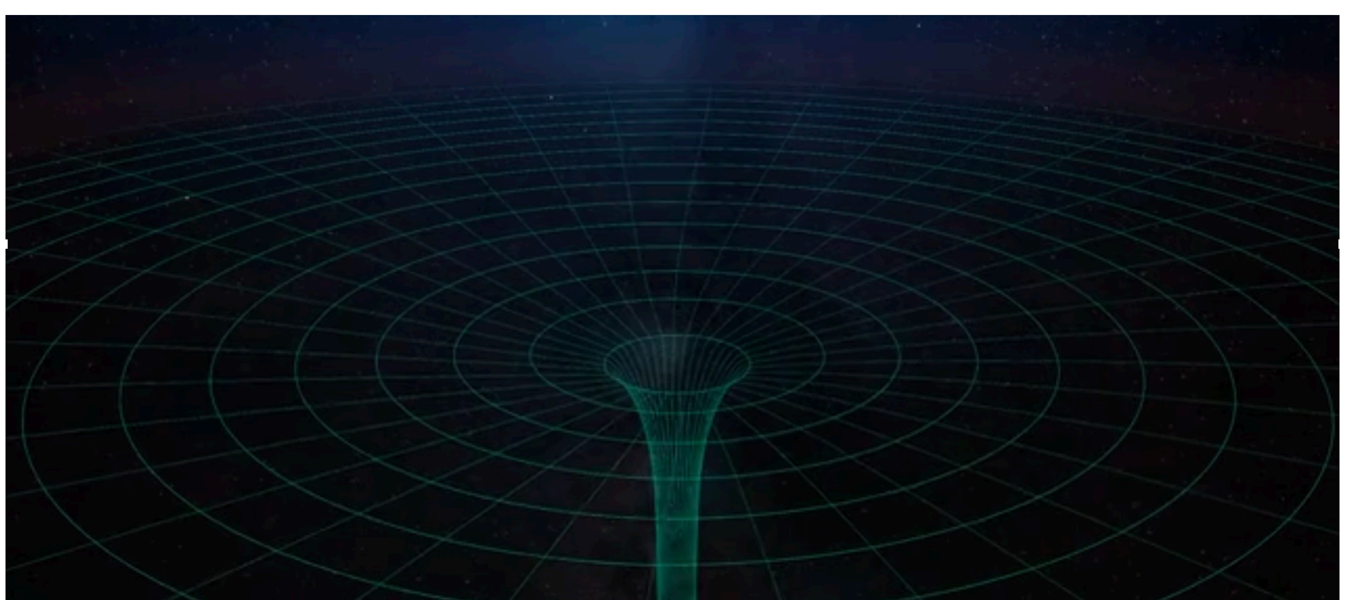

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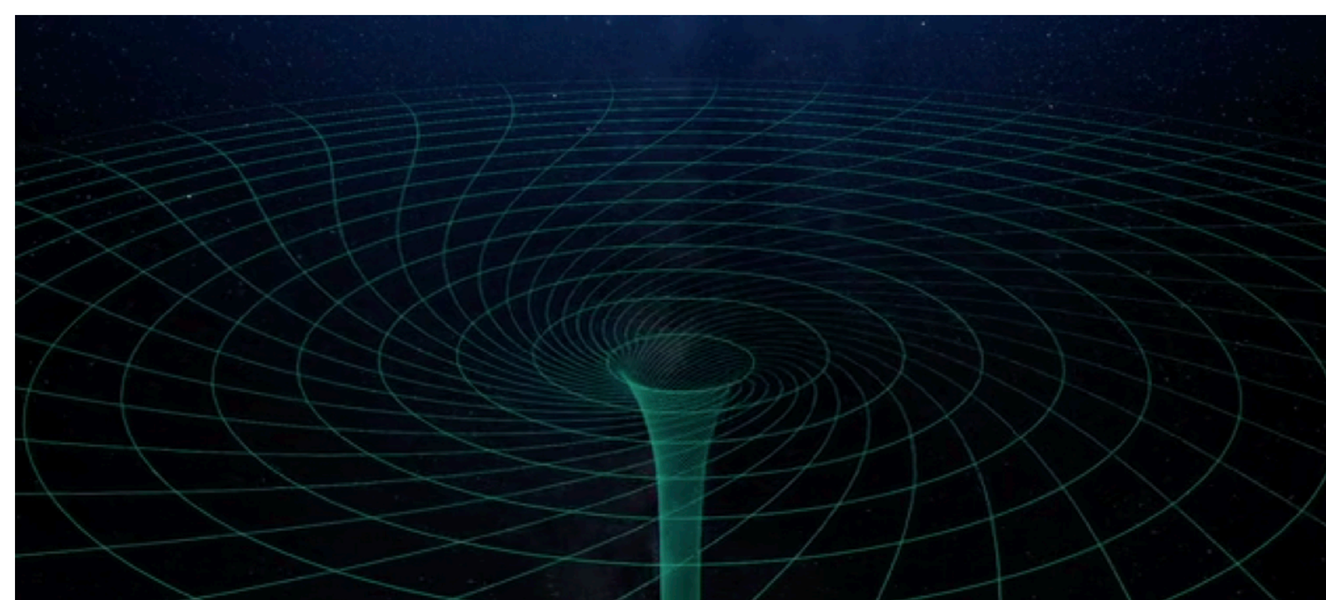

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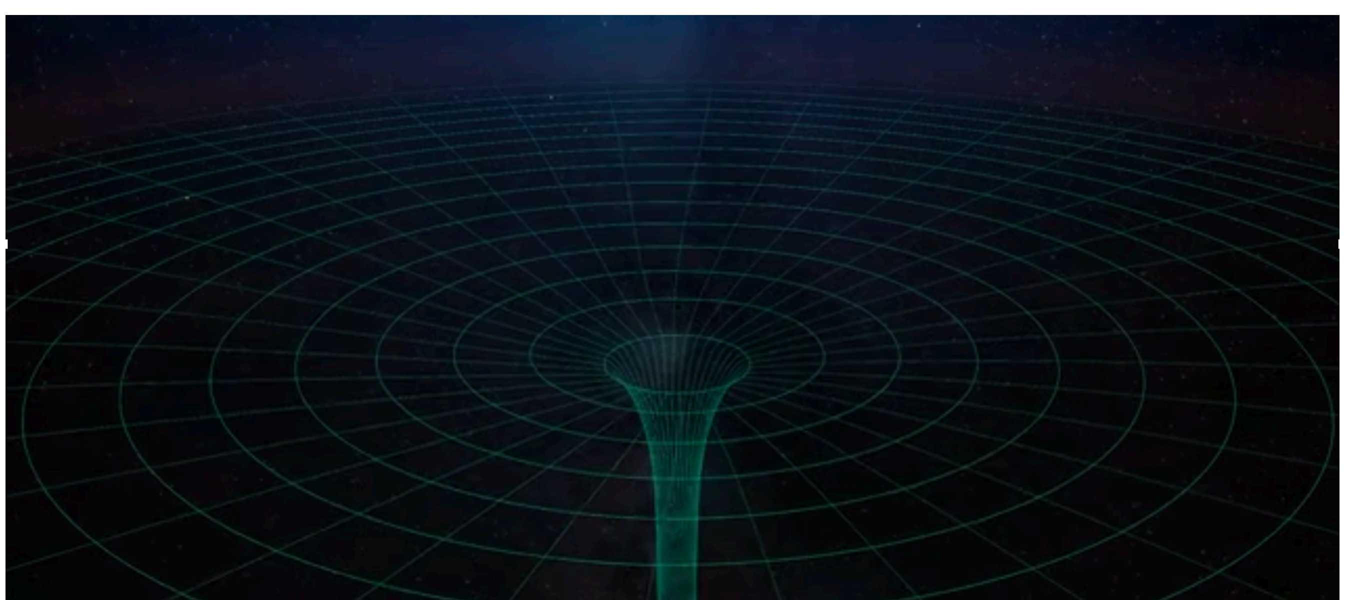
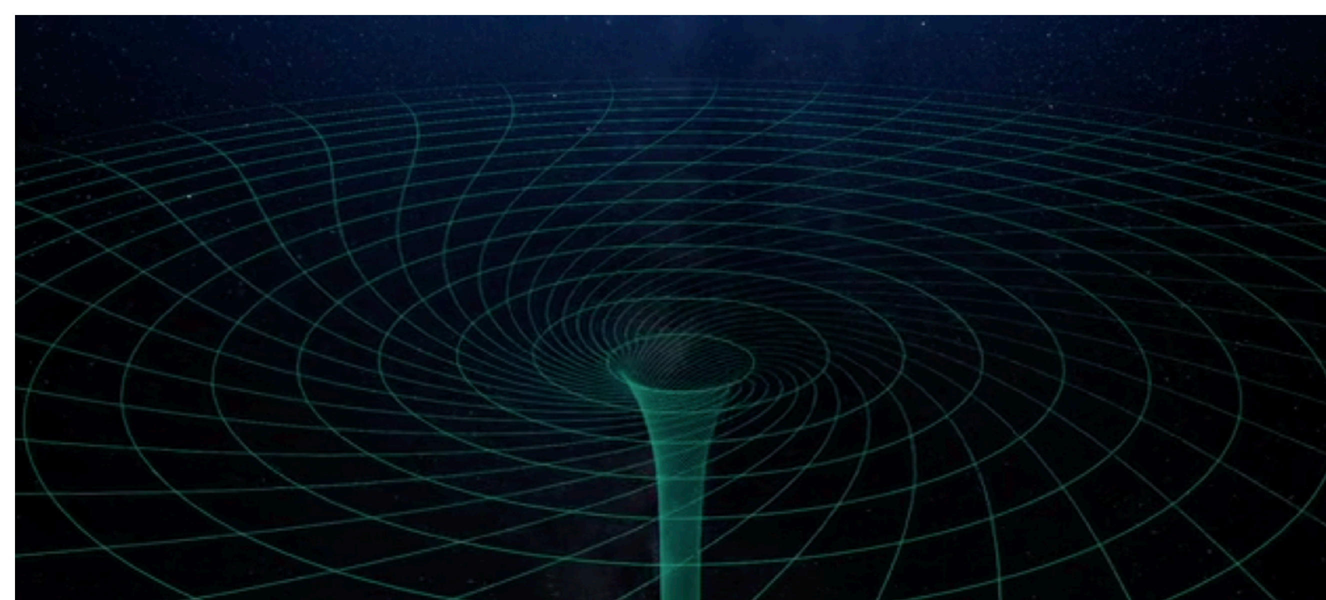
- Can think of this like a rotating floor giving angular momentum in one direction and taking in the other.

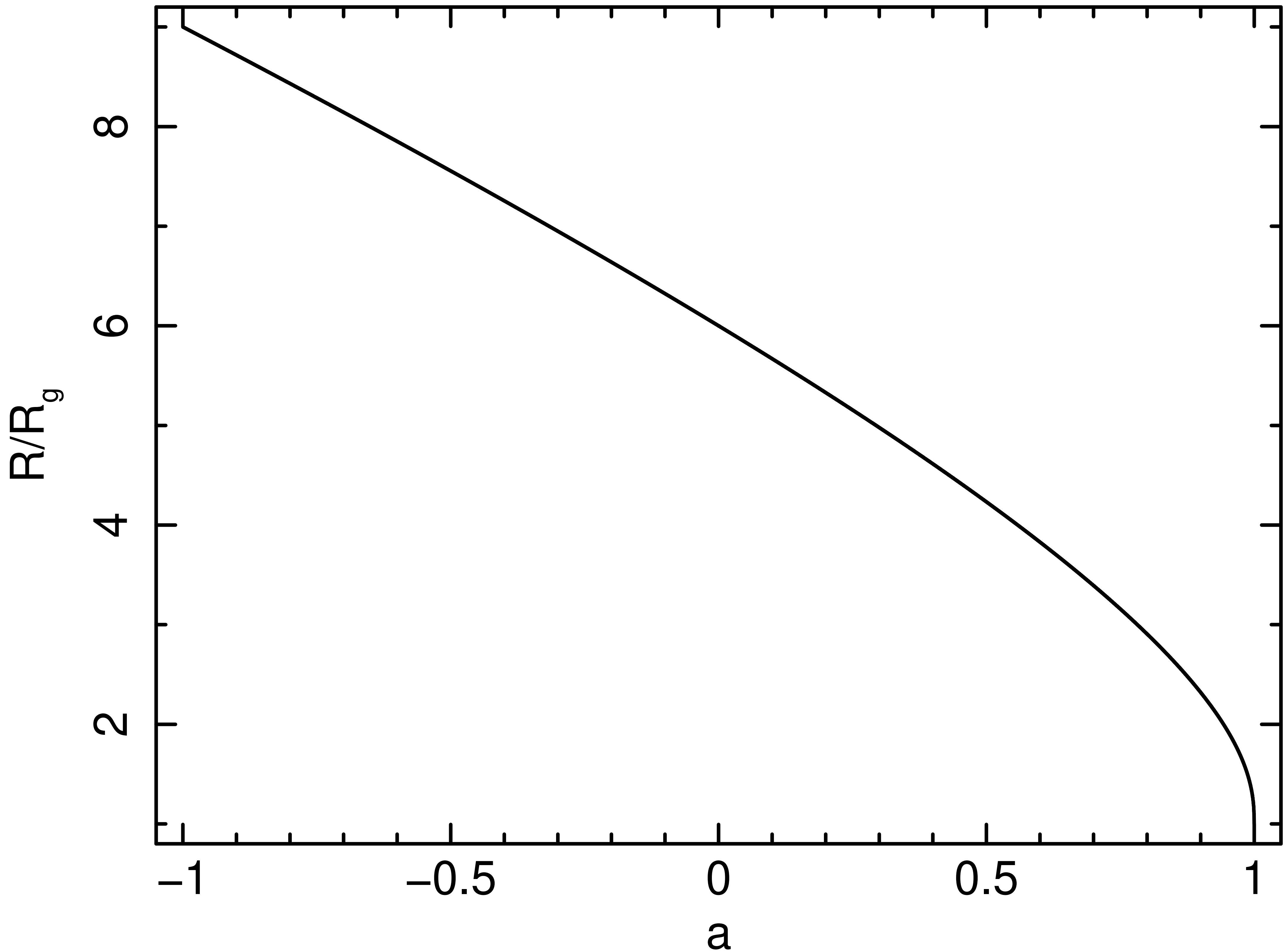
a = 0 **a > 0**





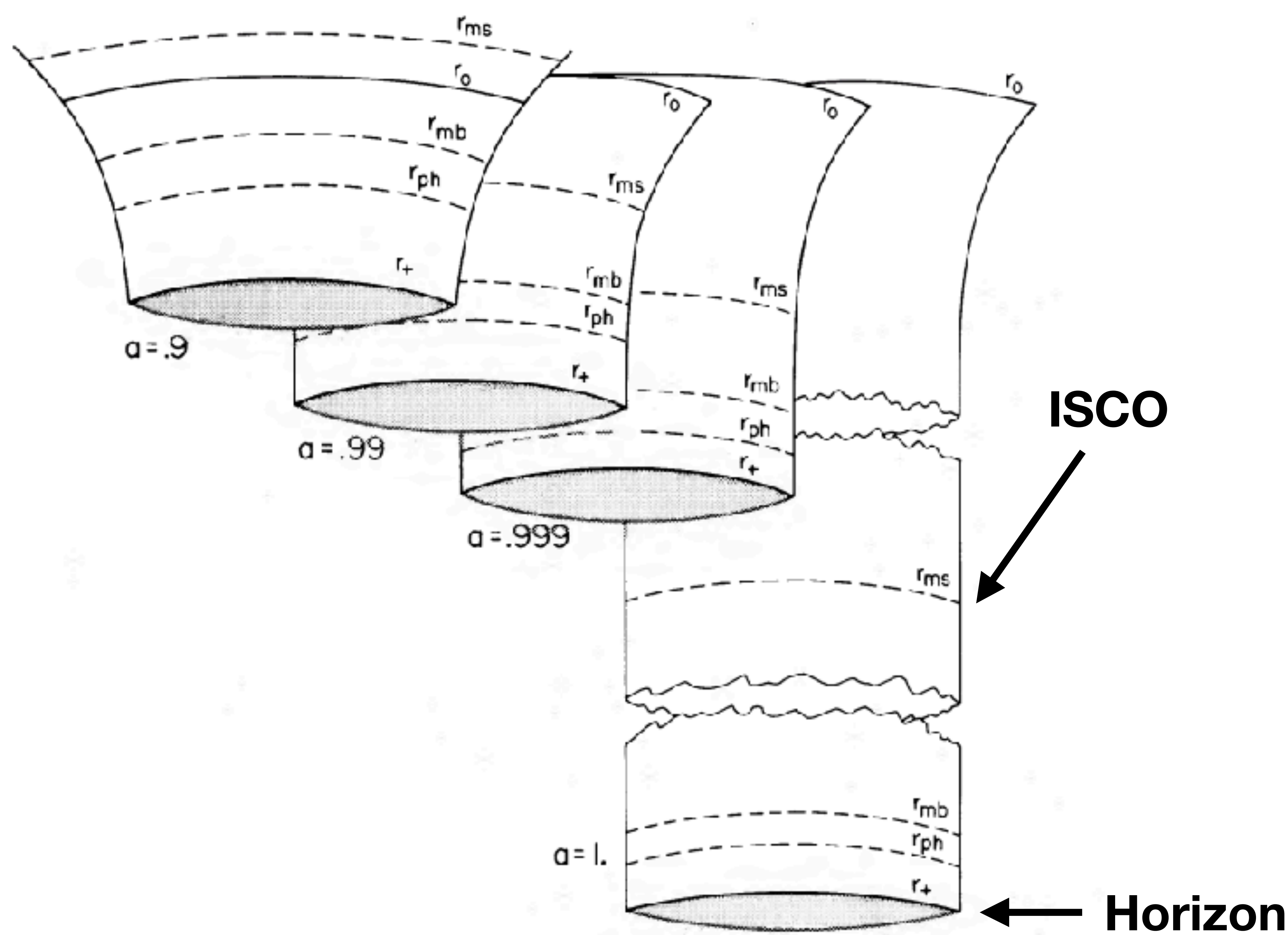
The Kerr Metric

- ISCO changes with spin:



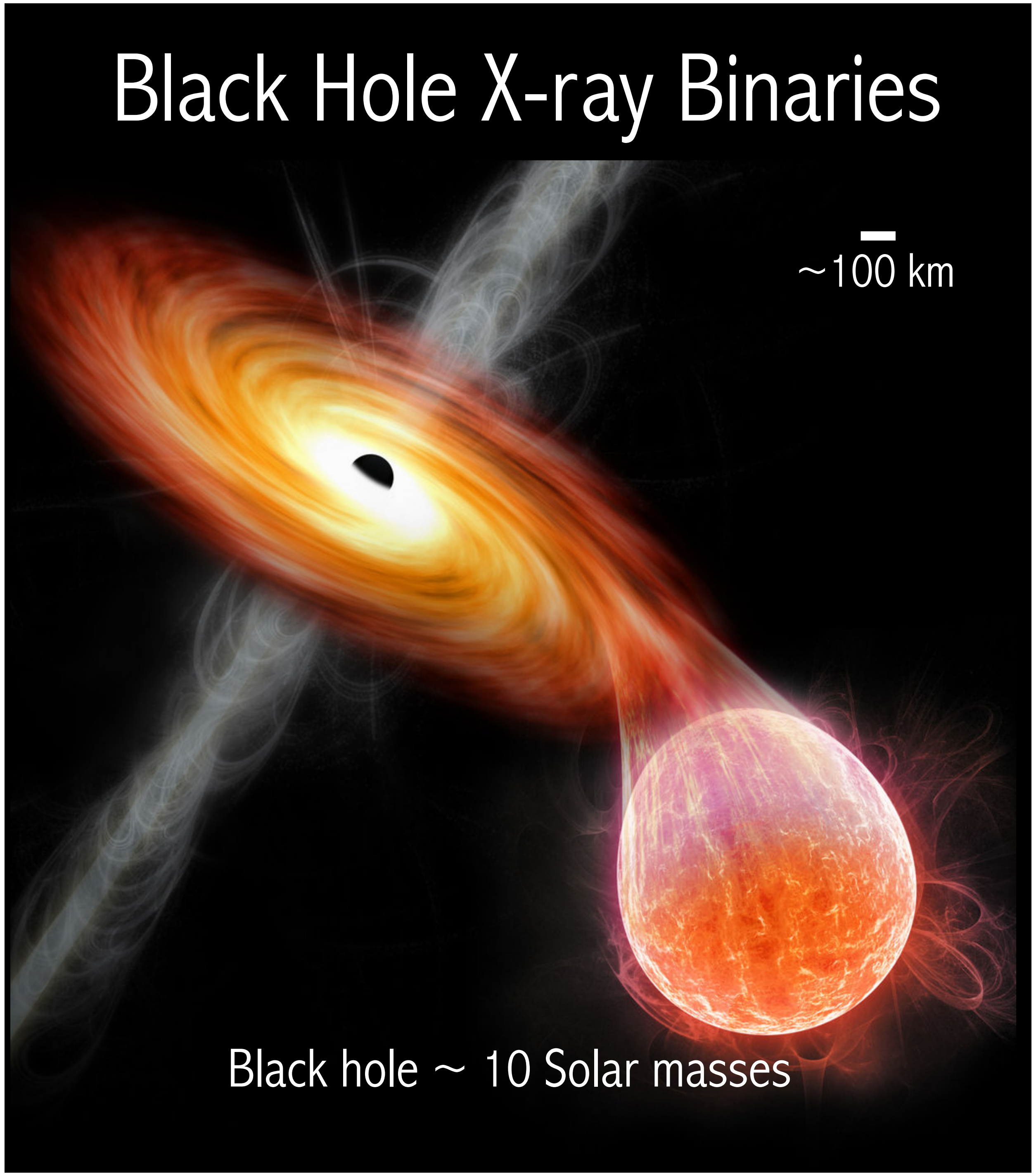
The Kerr Metric

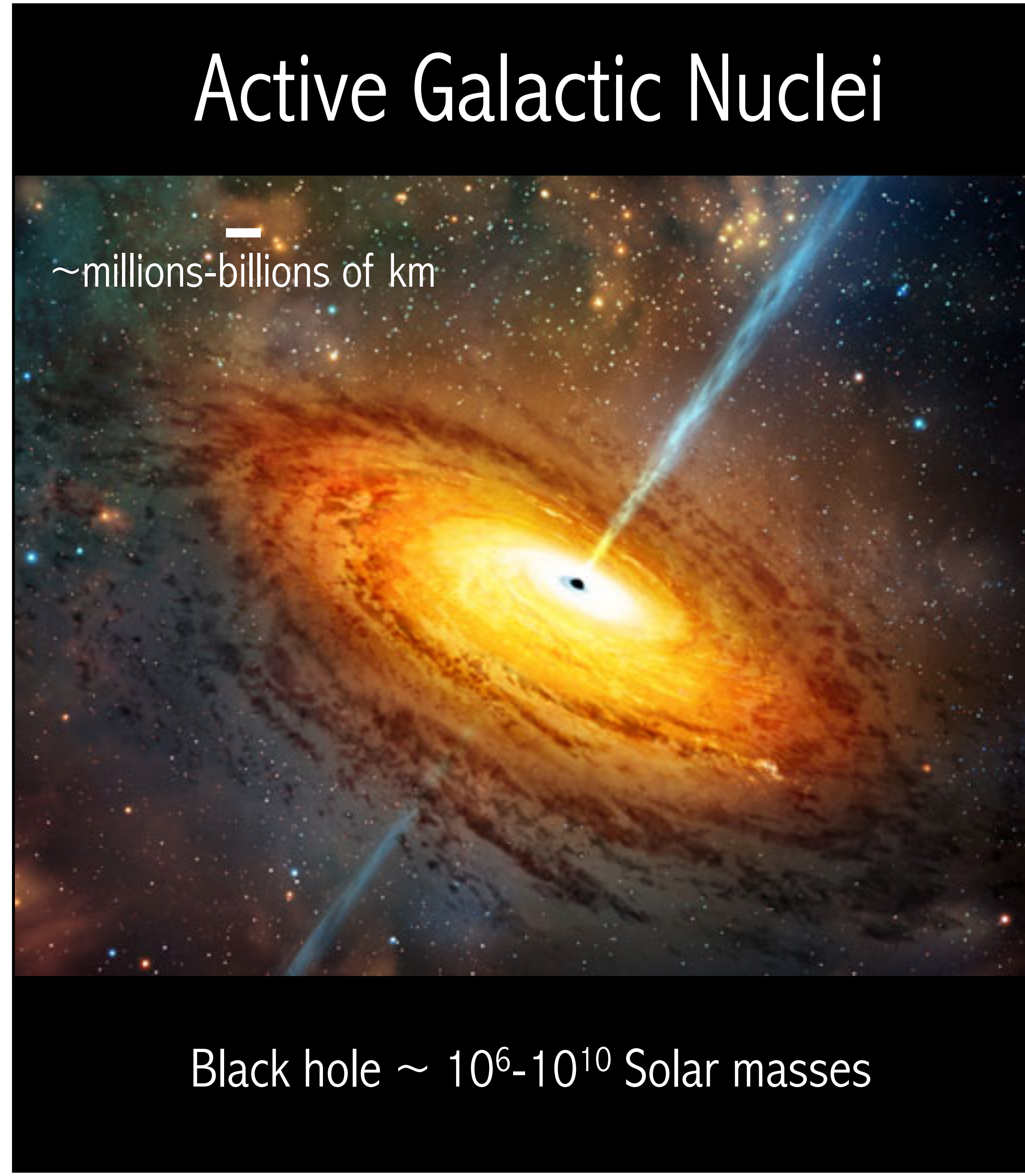
- What's going on at a=1? ISCO and horizon both at 1rg, but not in the same place! Break down of B-L coordinates.



Bardeen, Press & Teukolsky (1972)

Accreting Black Holes





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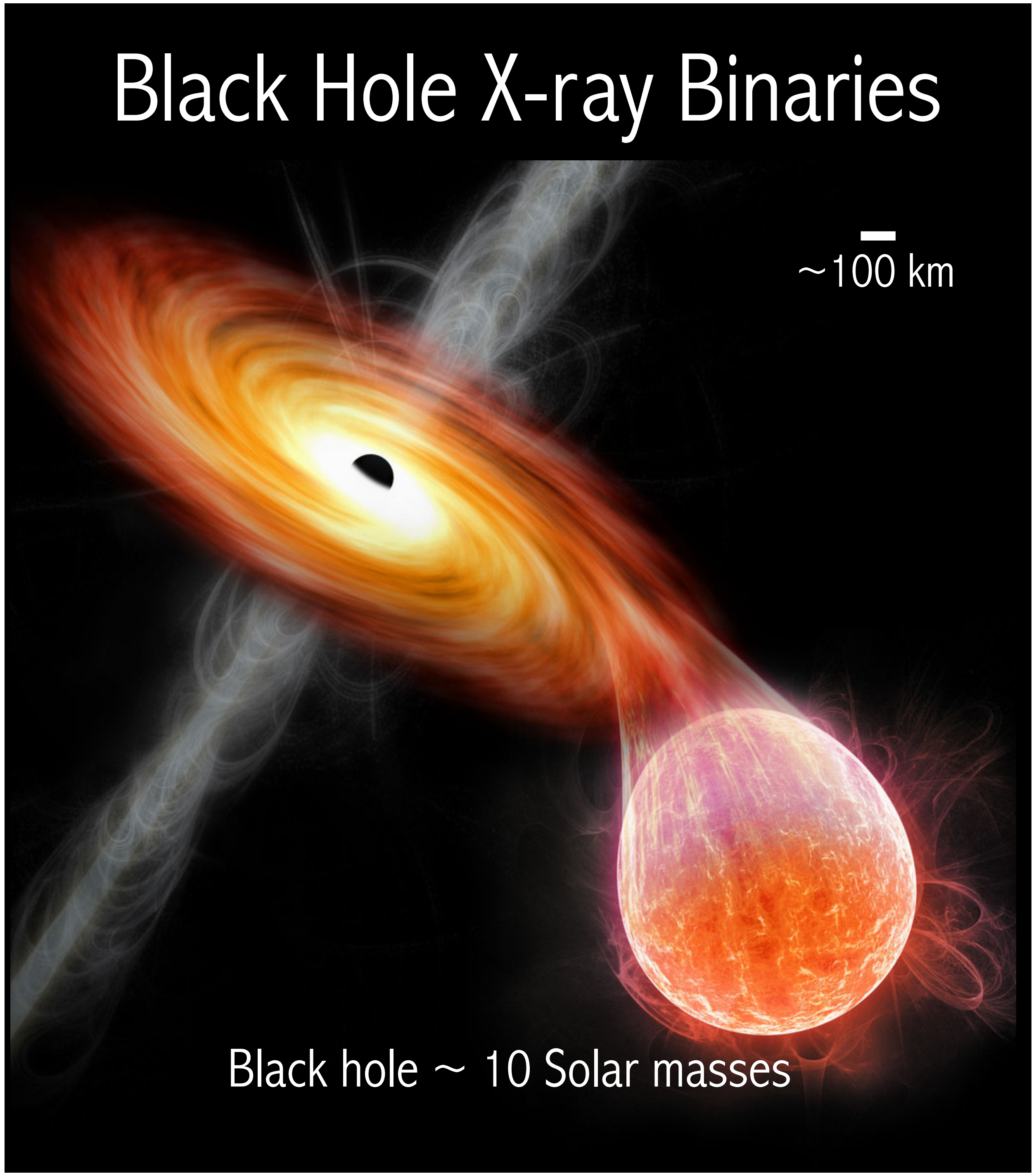

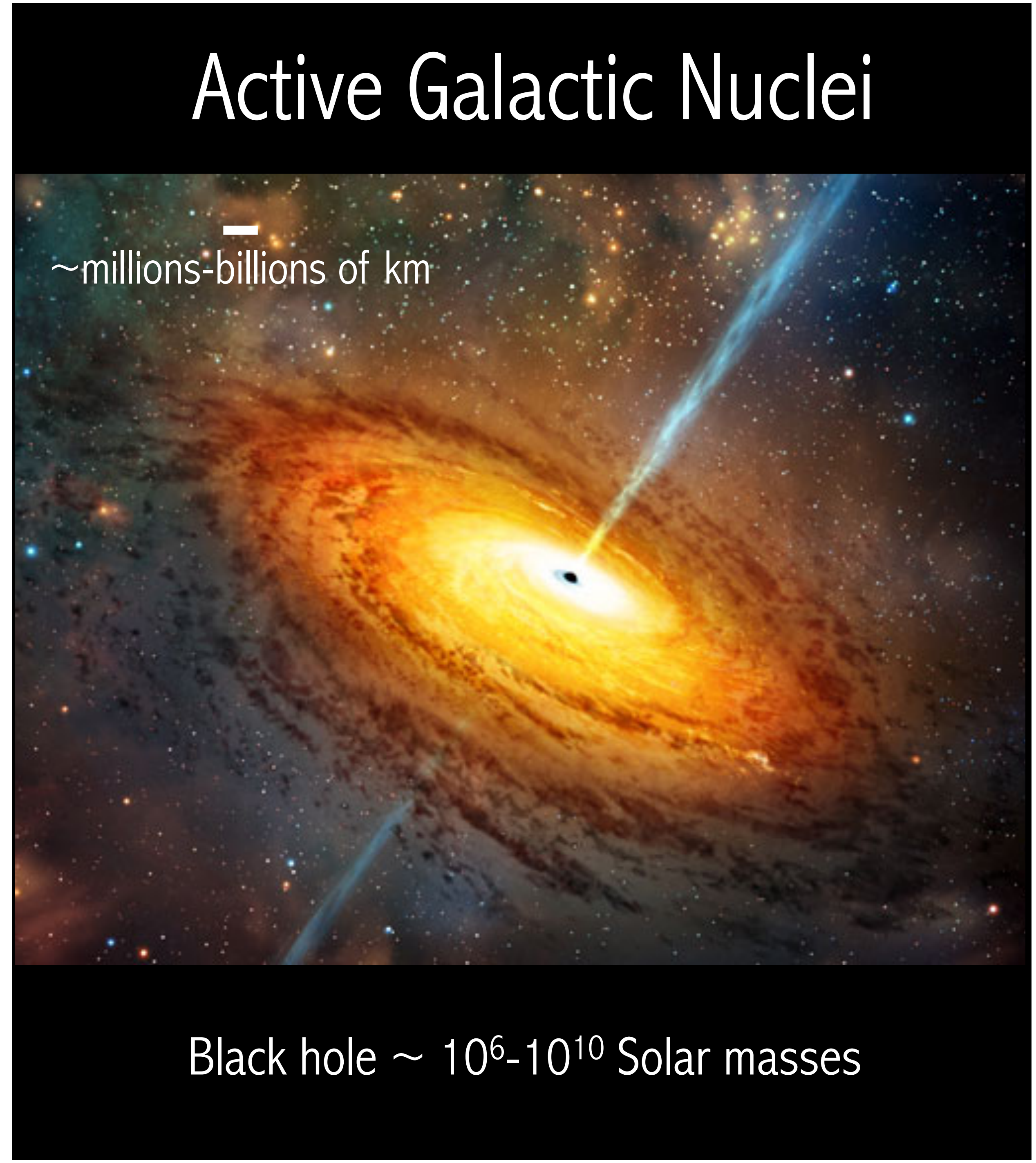

Power supply: gravitational potential energy of accreting material.

Therefore luminosity is the rest mass energy of accreted material multiplied by some efficiency factor: $L = \epsilon \dot{M} c^2$

Eddington Limit

Theoretical maximum luminosity when outwards radiation force balances inwards gravitational force. Above this luminosity, accreting material will be thrown off in winds.

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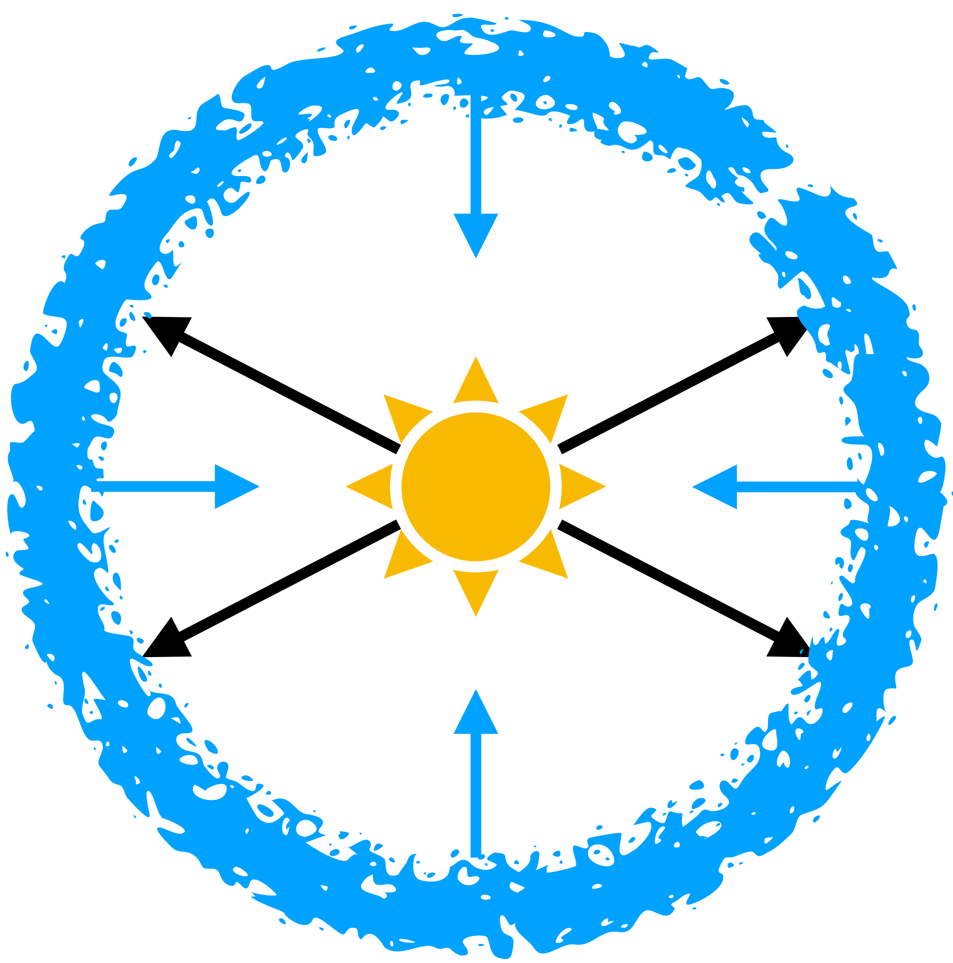
Assumptions:

- Spherical symmetry
- Dominant opacity = electron scattering (Thomson absorption cross-section).
- Gravity acts predominantly on protons.



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Neutral material therefore can set the two equal for the Eddington limit:

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Therefore can also define Eddington accretion rate:

$$\dot{M}_{\rm Edd} = \frac{4\pi GMm_p}{\epsilon c \sigma_T}$$

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Therefore AGN can be **much** more luminous than X-ray binaries after all!