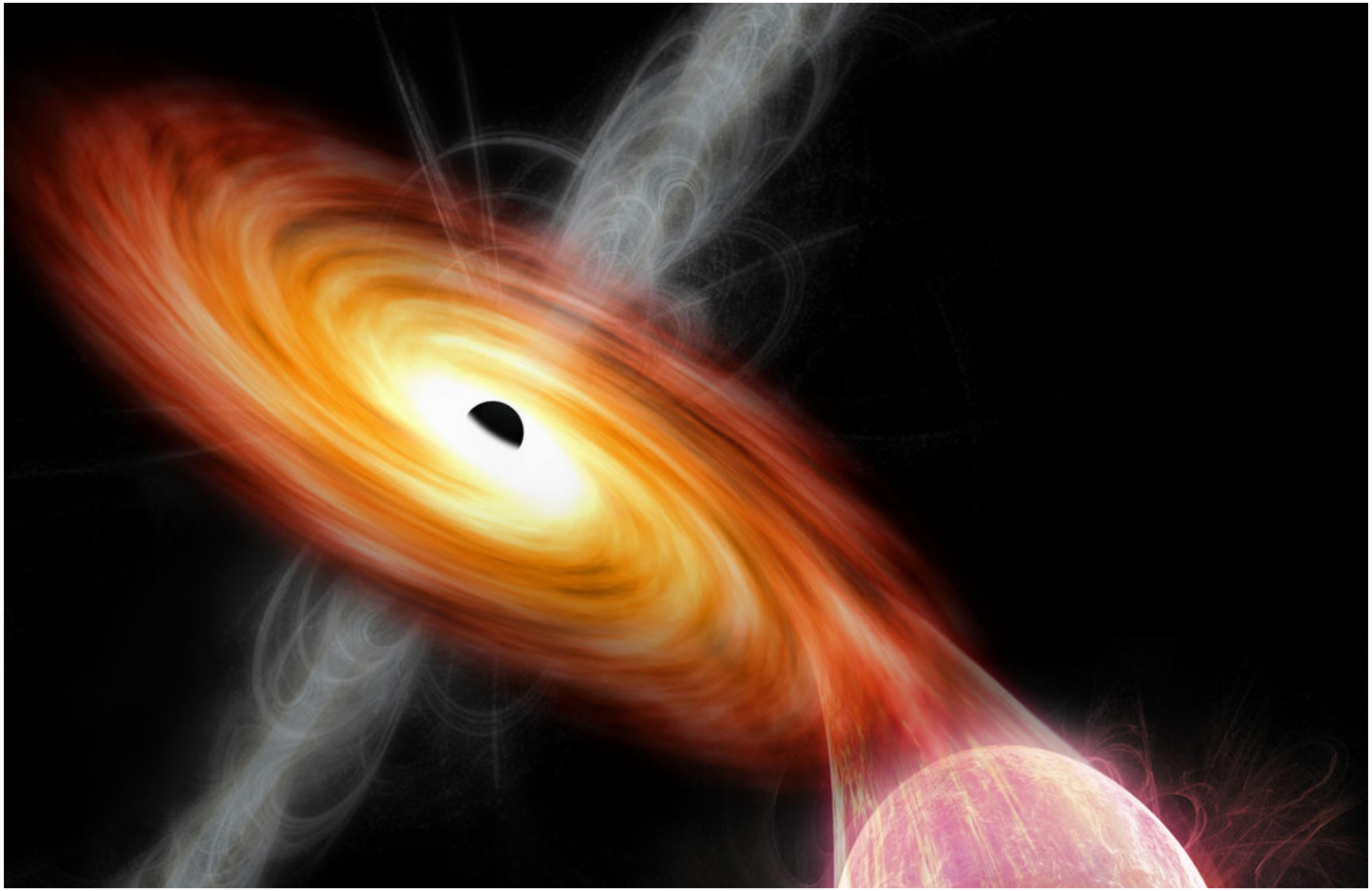
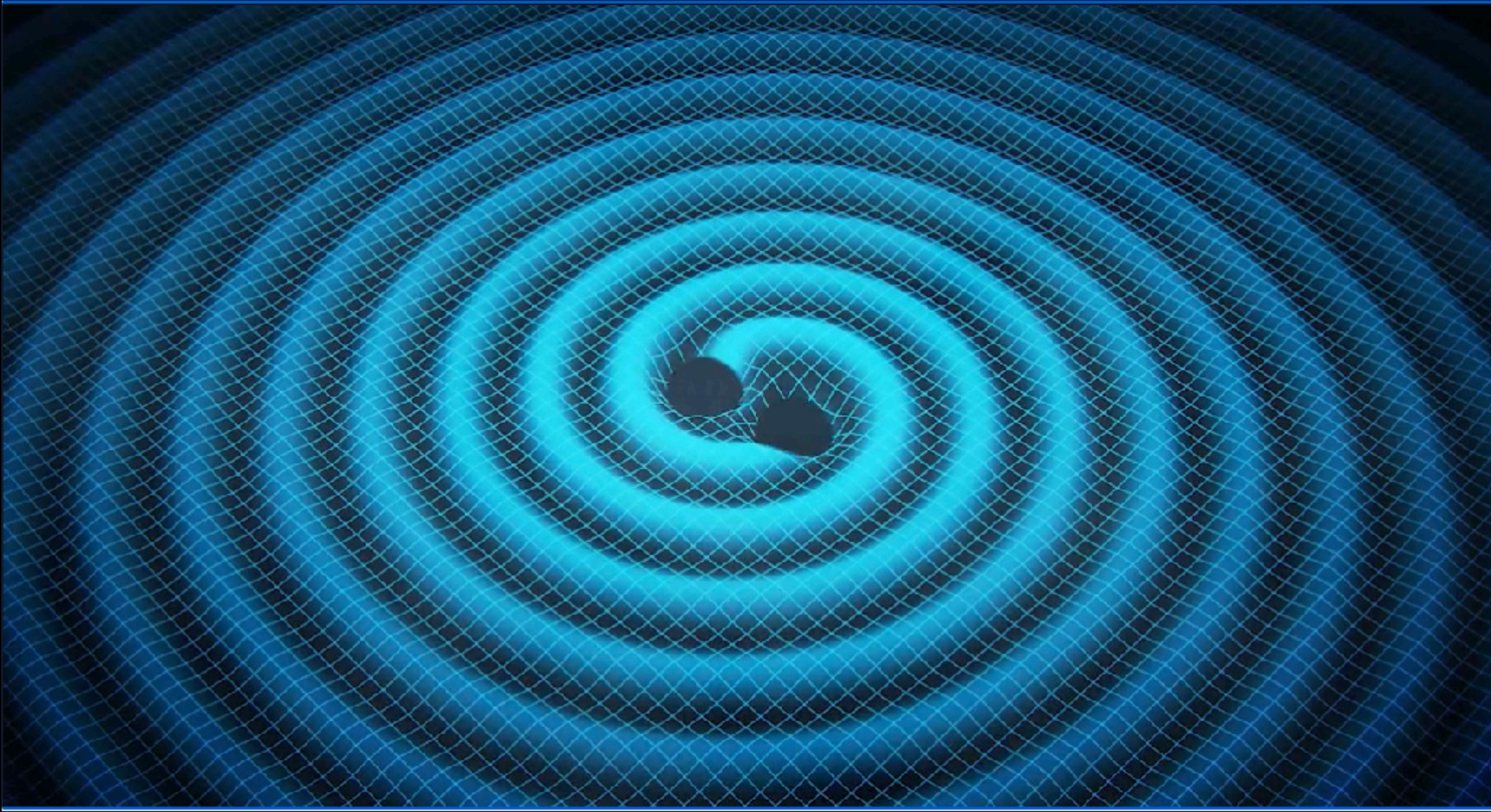
High Energy Astrophysics

Dr. Adam Ingram



Lecture 9
Gravitational Waves



Lecture 9
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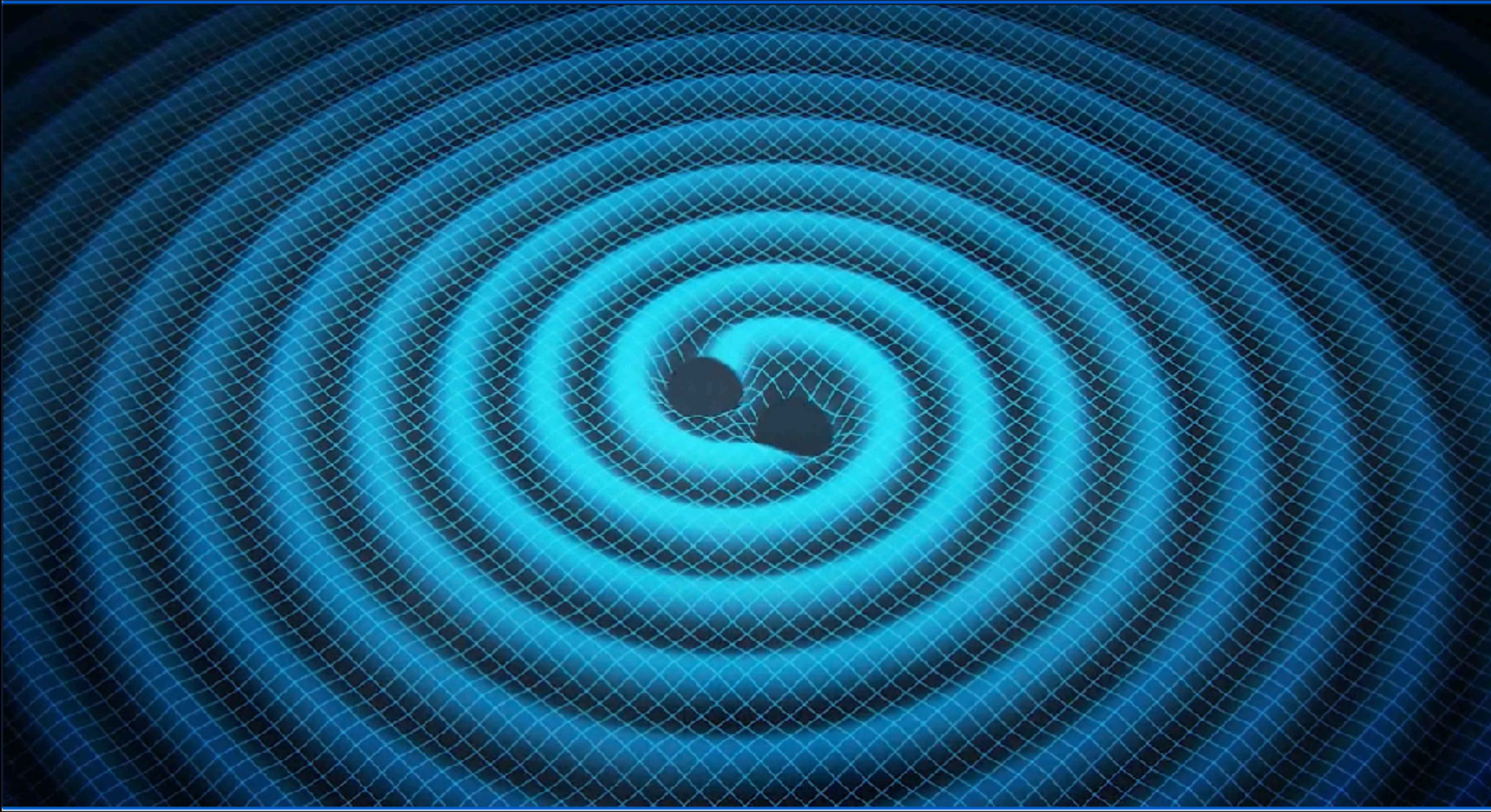
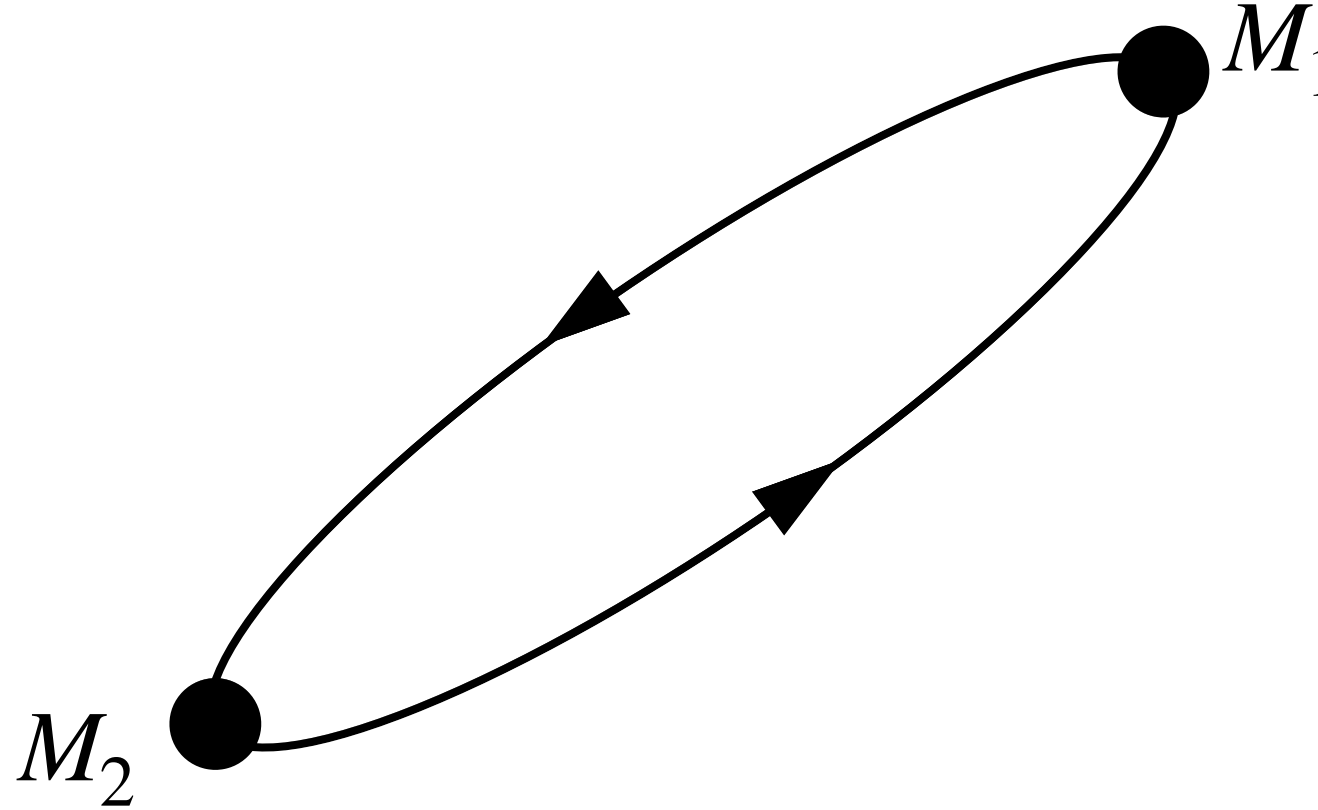


“Gravitational Wave Radiation by Binary Black Holes” by Ryan Rubenzahl
(https://rrubenza.github.io/project/p413_gws/RR_PHY413_GW_Paper.pdf)

Newtonian Gravitational Waves

- GWs are obviously an inherently relativistic phenomenon.
- BUT we can learn a lot by thinking about the Newtonian gravitational field we feel from a binary BH system.
- The gravitational field changes ever-so-slightly with binary orbital phase.
- Similarities to GWs: the changes in gravitational potential depend on the **quadrupole moment**, and the frequency of the changes is **twice the orbital phase**.
- The latter is obvious: we feel the same gravitational force when BH1 is on the left and BH2 is on the right as we do when BH1 is on the right and BH2 is on the left.



Orbital phase:

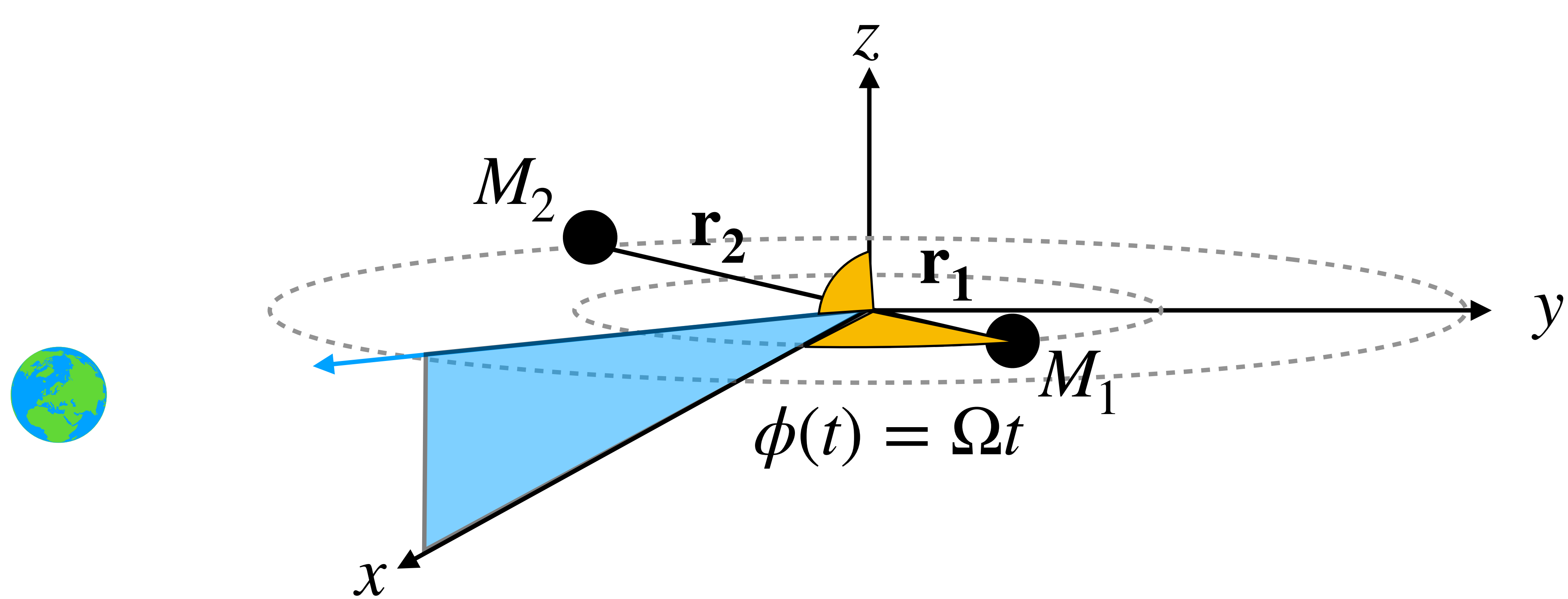
$$\phi(t) = \Omega t$$

Newtonian Gravitational Waves

- Binary system in x-y plane centred on CoM.
- Phase of BH1 is ϕ, phase of BH2 is ϕ+π.

$$\mathbf{r_1} = r_1(\cos\phi, \sin\phi, 0) \qquad \mathbf{r_2} = -r_2(\cos\phi, \sin\phi, 0)$$

- Total mass: $M = M_1 + M_2$; seperation: $r_a = r_1 + r_2$



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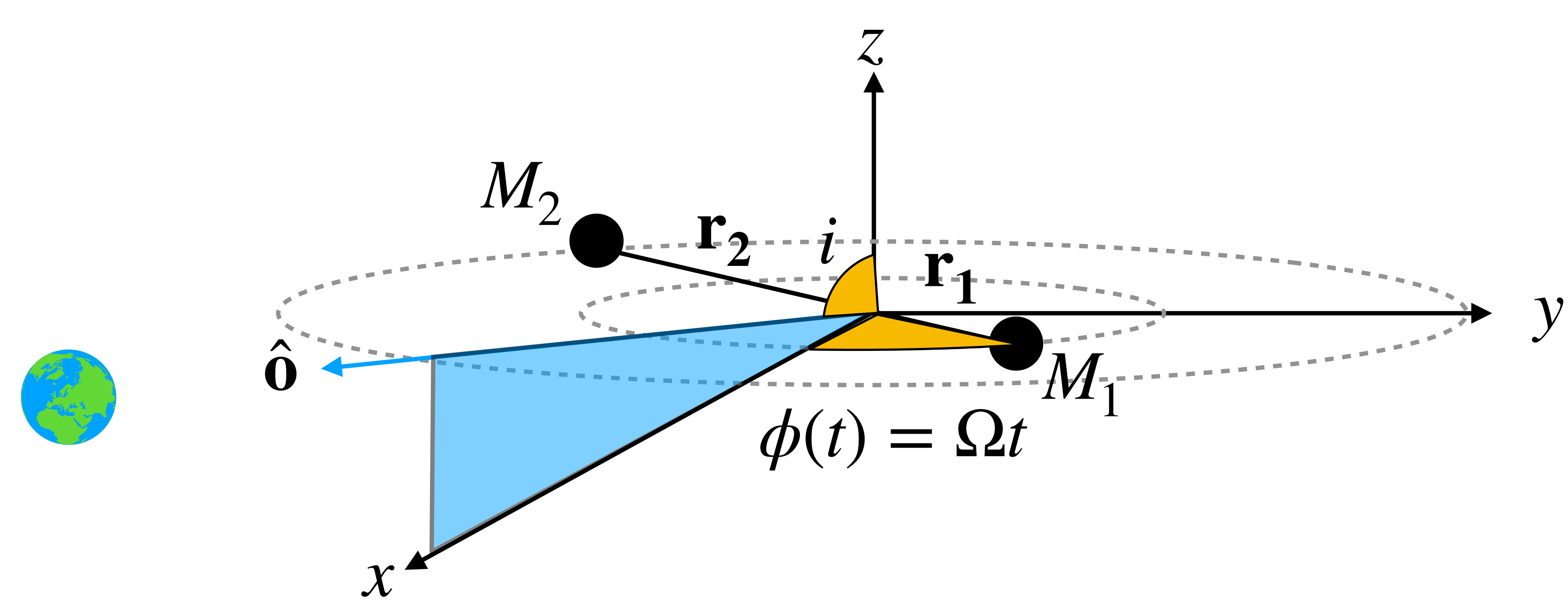
- Total mass: $M = M_1 + M_2$; seperation: $r_a = r_1 + r_2$
- Observer a distance r from the CoM along vector: $\hat{\mathbf{o}} = (\sin i, 0, \cos i)$
- Vector pointing from BH1 to observer: $\zeta_1 = -\mathbf{r_1} + r\hat{\mathbf{o}}$
- Vector pointing from BH2 to observer: $\zeta_2 = -\mathbf{r_2} + r\hat{\mathbf{o}}$



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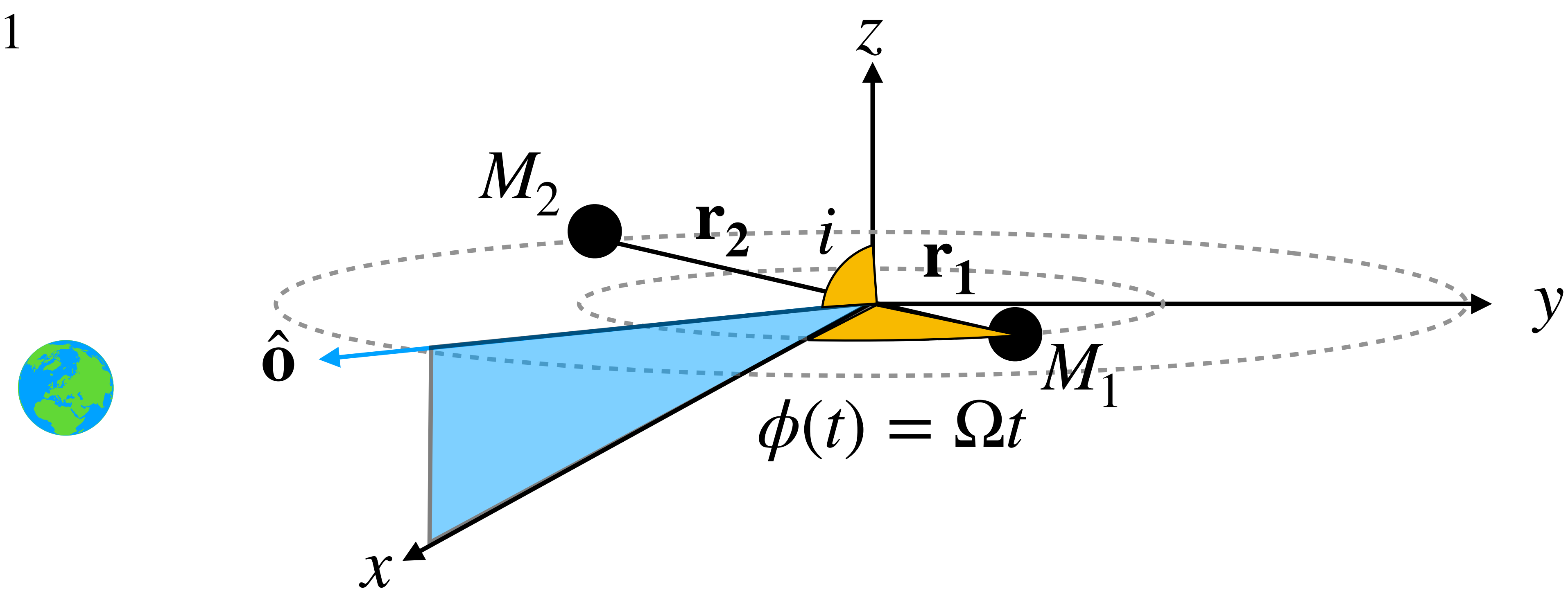
- Gravitational force on observer (mass m) from BH1 in direction of CoM:

$$F_1 = \frac{GM_1 m}{\zeta_1^2}\frac{\zeta_{\mathbf{1}} \cdot \hat{\mathbf{o}}}{\zeta_1}$$



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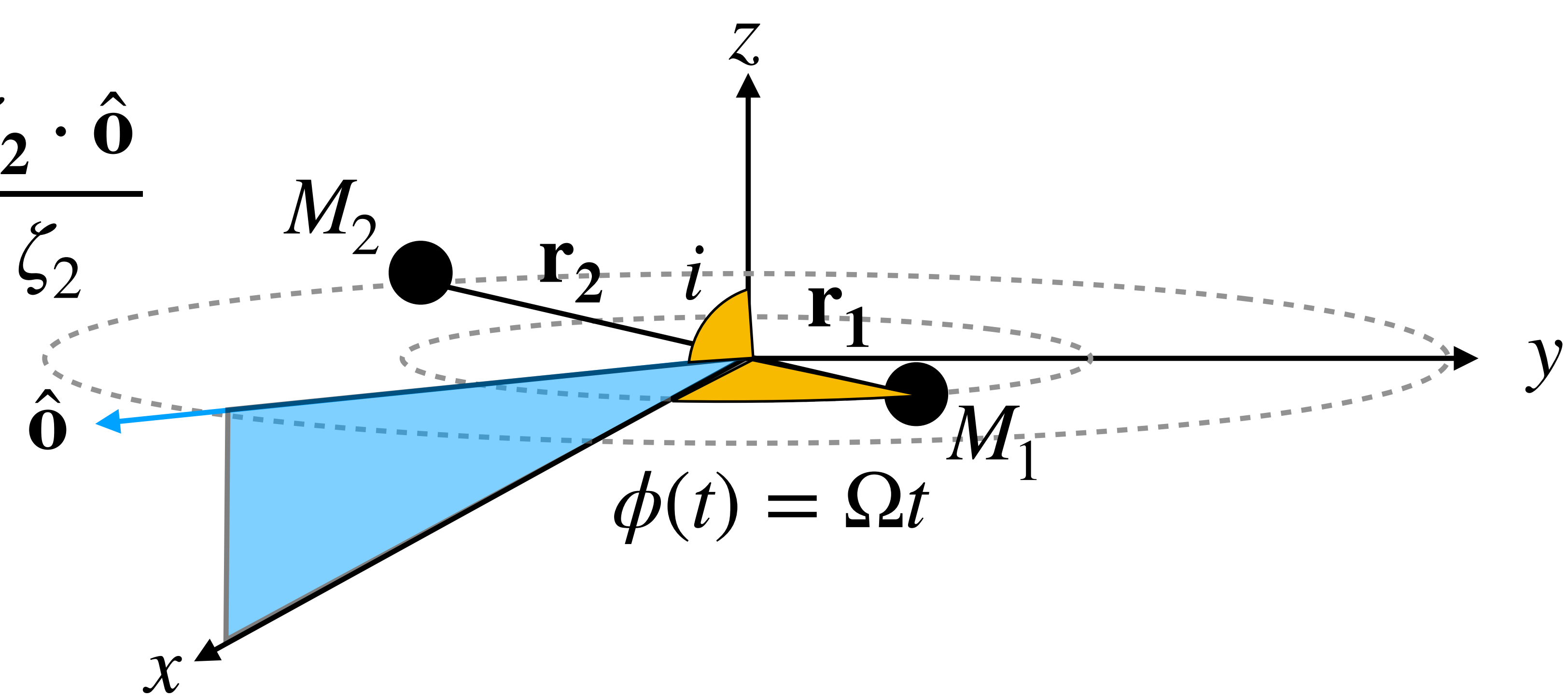
- From BH2: $F_2 = \frac{GM_2 m}{\zeta_2^2}\frac{\zeta_\mathbf{2}\cdot\hat{\mathbf{o}}}{\zeta_2}$



Newtonian Gravitational Waves

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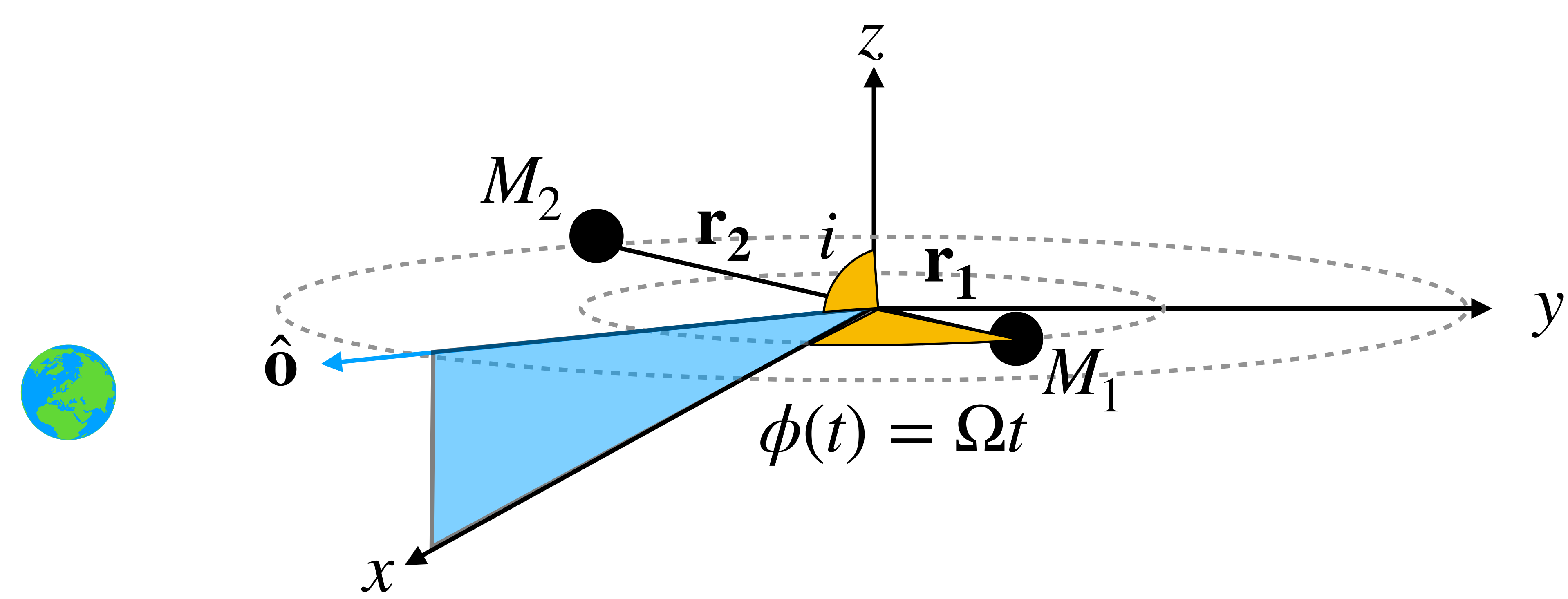
$$\mathbf{r_1} = r_1(\cos\phi, \sin\phi, 0) \qquad \mathbf{r_2} = -r_2(\cos\phi, \sin\phi, 0)$$

- Total gravitational force in direction of CoM:

$$F = \frac{GM_1m\ (r - r_1\cos\phi\sin i)}{(r^2 + r_1^2 - 2r_1r\cos\phi\sin i)^{3/2}} + \frac{GM_2m\ (r + r_2\cos\phi\sin i)}{(r^2 + r_2^2 + 2r_2r\cos\phi\sin i)^{3/2}}$$



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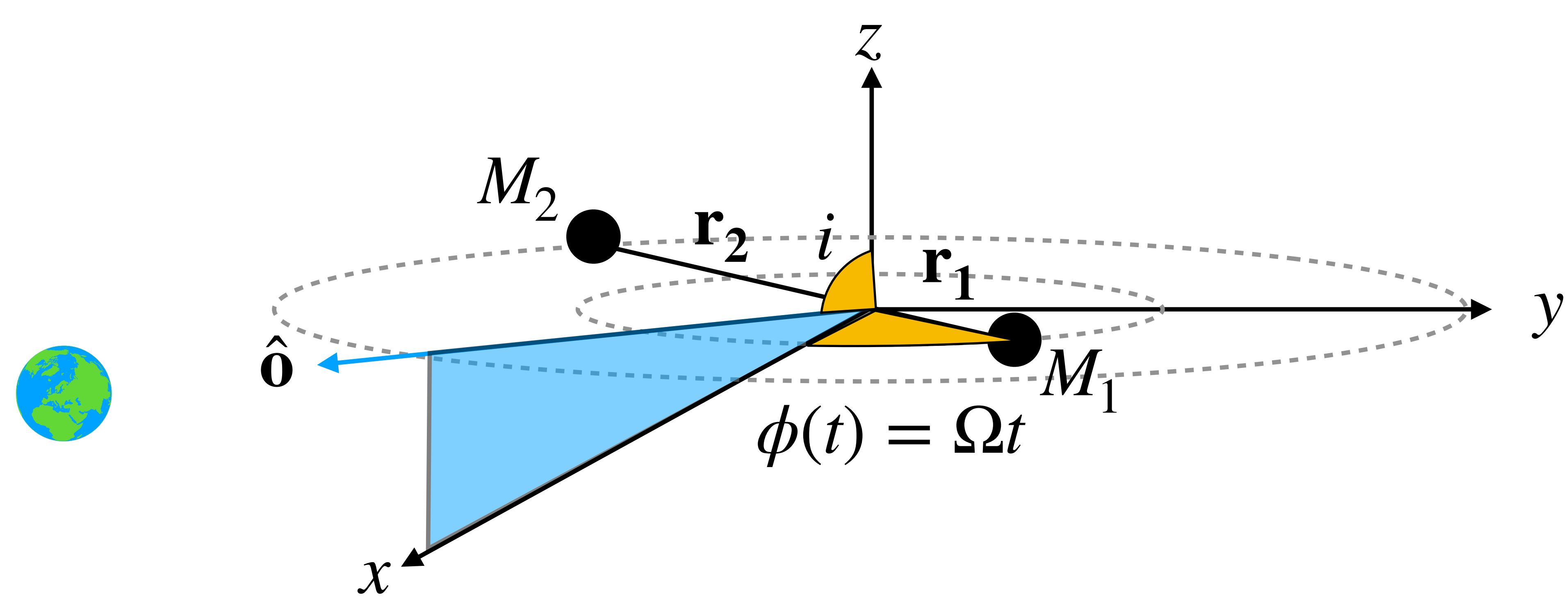
- $r \gg r_a \implies$ Binomial expansion:



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- $r \gg r_a \implies$ Binomial expansion:

$$F \approx \frac{GM_1 m}{r^3}\ (r - r_1\cos\phi\sin i)\left[1 - \frac{3}{2}\left(\frac{r_1}{r}\right)^2 + 3\left(\frac{r_1}{r}\right)\cos\phi\sin i\right]$$

$$+\frac{GM_2 m}{r^3}\ (r + r_2\cos\phi\sin i)\left[1 - \frac{3}{2}\left(\frac{r_2}{r}\right)^2 - 3\left(\frac{r_2}{r}\right)\cos\phi\sin i\right]$$

Newtonian Gravitational Waves

- Simplify to equal mass binary

$$F \approx \frac{GM_1 m}{r^3}\,(r - r_1\cos\phi\sin i)\left[1 - \frac{3}{2}\left(\frac{r_1}{r}\right)^2 + 3\left(\frac{r_1}{r}\right)\cos\phi\sin i\right]$$

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$$F \approx \frac{GM_1 m}{r^3}\left[2r - 3\frac{r_1^2}{r} - 6\frac{r_1^2}{r}\cos^2\phi\sin^2 i\right]$$

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Monopole term

Quadrupole term

Newtonian Gravitational Waves

- Express in terms of gravitational potential:

$$V(r,t) \approx -\frac{GM}{r} + \frac{GM}{2r^3}\ r_1^2\ \left[1 + \sin^2 i \left(1 + \cos(2\omega t)\right)\right]$$

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- Can write quadrupole potential in terms of the **quadrupole moment tensor**:

$$V_q(r,t) = \frac{G}{r^3}\left\{\sum_{i,j} Q_{ij}\ \hat{o}_i\ \hat{o}_j - \frac{5}{2}Q_{33}\right\}$$

$$\hat{o}_1 = \sin i$$
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$$\hat{o}_1 = \sin i \qquad \hat{o}_2 = 0 \qquad \hat{o}_3 = \cos i$$

- Where:

$$Q_{ij} = \int \rho(\mathbf{r})\ \left(r_i r_j - \frac{1}{3}|\mathbf{r}|^2 \delta_{ij}\right)\ d^3\mathbf{r}$$

- Which in our case is:

$$Q_{ij} = M_1\left(r_{1,i}r_{1,j} - \frac{r_1^2}{3}\delta_{ij}\right) + M_2\left(r_{2,i}r_{2,j} - \frac{r_2^2}{3}\delta_{ij}\right)$$

Newtonian Gravitational Waves

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$$\mathbf{r_1} = r_1 (\cos\phi, \sin\phi, 0) \qquad \mathbf{r_2} = -r_2 (\cos\phi, \sin\phi, 0)$$

- For example:

$$Q_{11} = M_1 \left[r_1^2 \cos^2\phi - \frac{r_1^2}{3} \right] + M_2 \left[r_2^2 \cos^2\phi - \frac{r_2^2}{3} \right]$$

Newtonian Gravitational Waves

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- $M_1 = M_2 \implies$

$$Q_{11} = 2M_1 r_1^2 \left[\cos^2\phi - \frac{1}{3} \right] = \frac{M r_a^2}{2} \frac{1}{4} \left[\frac{1}{3} + \cos(2\phi) \right]$$

Newtonian Gravitational Waves

$$Q_{ij} = M_1\left(r_{1,i}r_{1,j} - \frac{r_1^2}{3}\delta_{ij}\right) + M_2\left(r_{2,i}r_{2,j} - \frac{r_2^2}{3}\delta_{ij}\right)$$

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- $M_1 = M_2 \implies$

$$Q_{11} = 2M_1 r_1^2\left[\cos^2\phi - \frac{1}{3}\right] = \frac{Mr_a^2}{2}\frac{1}{4}\left[\frac{1}{3} + \cos(2\phi)\right]$$

- $M_1 > M_2 \implies$

$$Q_{11} = \frac{Mr_a^2}{2}\mu\left[\frac{1}{3} + \cos(2\phi)\right]$$

Symmetric mass:

$$\mu \equiv \frac{M_1M_2}{(M_1+M_2)^2}$$

Newtonian Gravitational Waves

$$Q_{ij} = M_1 \left(r_{1,i} r_{1,j} - \frac{r_1^2}{3} \delta_{ij} \right) + M_2 \left(r_{2,i} r_{2,j} - \frac{r_2^2}{3} \delta_{ij} \right)$$

$$\mathbf{r_1} = r_1 (\cos\phi, \sin\phi, 0) \qquad \mathbf{r_2} = -r_2 (\cos\phi, \sin\phi, 0)$$

- Quadrupole moment tensor for a binary BH system:

$$Q_{ij} = \frac{1}{2} \mu M r_a^2 \begin{pmatrix} 1/3 + \cos(2\phi) & \sin(2\phi) & 0 \\ \sin(2\phi) & 1/3 - \cos(2\phi) & 0 \\ 0 & 0 & -2/3 \end{pmatrix}$$

Newtonian Gravitational Waves

$$V_q(r,t) = \frac{G}{r^3}\left\{\sum_{i,j} Q_{ij}\ \hat{o}_i\ \hat{o}_j - \frac{5}{2}Q_{33}\right\}$$

$$\hat{o}_1 = \sin i$$
$$\hat{o}_2 = 0$$
$$\hat{o}_3 = \cos i$$

- Only non-zero terms are $Q_{11}o_1o_1$ and $Q_{33}o_3o_3$:

$$V_q(r,t) = \frac{G}{3r^3}\left\{Q_{11}\sin^2 i + Q_{33}\cos^2 i - \frac{5}{2}Q_{33}\right\}$$

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- For equal mass binary, end up with:

$$V_q(r,t) = \frac{GM}{2r^3}\ r_1^2\ \left\{1 + \sin^2 i[1 + \cos(2\Omega t)]\right\}$$

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Newtonian Gravitational Waves

So even in Newtonian gravity we experience a changing gravitational force from a binary system due to a changing **gravitational quadrupole moment** with frequency 2Ω.

But of course there are many important differences:

- GWs are **ripples in spacetime** due to the changing quadrupole moment, not action-at-a-distance changes in gravitational field.
- Causality: GWs **propagate** at the speed of light.
- GWs are tiny, but **much** bigger than the ludicrously tiny effect of a changing Newtonian gravitational field we've explored so far.

GWs from the Einstein Equations

$$G_{\mu\nu} = 8\pi \frac{G}{c^4} T_{\mu\nu}$$

Einstein tensor
= spacetime curvature
= the metric independent of coordinate system

Stress-energy tensor
= mass density and pressure
In SR, this is:

$$T_{\mu\nu} = \begin{pmatrix} \rho_0 c^2 & 0 & 0 & 0 \\ 0 & P_x & 0 & 0 \\ 0 & 0 & P_y & 0 \\ 0 & 0 & 0 & P_z \end{pmatrix}$$

GWs from the Einstein Equations

$$G_{\mu\nu} = 8\pi \frac{G}{c^4} T_{\mu\nu}$$

Write metric as Minkowski + small perturbation:

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \qquad\qquad h_{\mu\nu} \ll 1$$

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After a lot of maths, and the right choice of coordinates (gauge), the Einstein equations become:

$$\left(\nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) h_{\mu\nu} = -16\pi \frac{G}{c^4} T_{\mu\nu}$$

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In empty space (T=0), we therefore get:

$$\nabla^2 h_{\mu\nu} = \frac{1}{c^2} \frac{\partial^2 h_{\mu\nu}}{\partial t^2}$$

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In empty space (T=0), we therefore get:

$$\nabla^2 h_{\mu\nu} = \frac{1}{c^2} \frac{\partial^2 h_{\mu\nu}}{\partial t^2}$$

…which is a wave equation!
Ripples in the metric propagate outwards from a disturbance at the speed of light!

GWs from the Einstein Equations

$$\nabla^2 h_{\mu\nu} = \frac{1}{c^2}\frac{\partial^2 h_{\mu\nu}}{\partial t^2}$$

Plane wave solution:

$$h_{\mu\nu} = A_{\mu\nu}\mathrm{e}^{ik_\alpha x^\alpha}$$

GWs from the Einstein Equations

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Plane wave solution:

$$h_{\mu\nu} = A_{\mu\nu} e^{ik_\alpha x^\alpha}$$

Set up coordinate system so that wave propagates in z-direction:

$$h_{\mu\nu} = A_{\mu\nu} e^{i(kz-\omega t)}$$

GWs from the Einstein Equations

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Plane wave solution:

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Set up coordinate system so that wave propagates in z-direction:

$$h_{\mu\nu} = A_{\mu\nu} \mathrm{e}^{i(kz-\omega t)}$$

Amplitude is linear sum of two modes:

$$A_{\mu\nu} = \underbrace{h_+ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}}_{\text{Plus mode}} + \underbrace{h_x \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}}_{\text{Cross mode}}$$

(Cartesian; 0=t, 1=x, 2=y, 3=z)

GW solutions

What do the waves actually look like?

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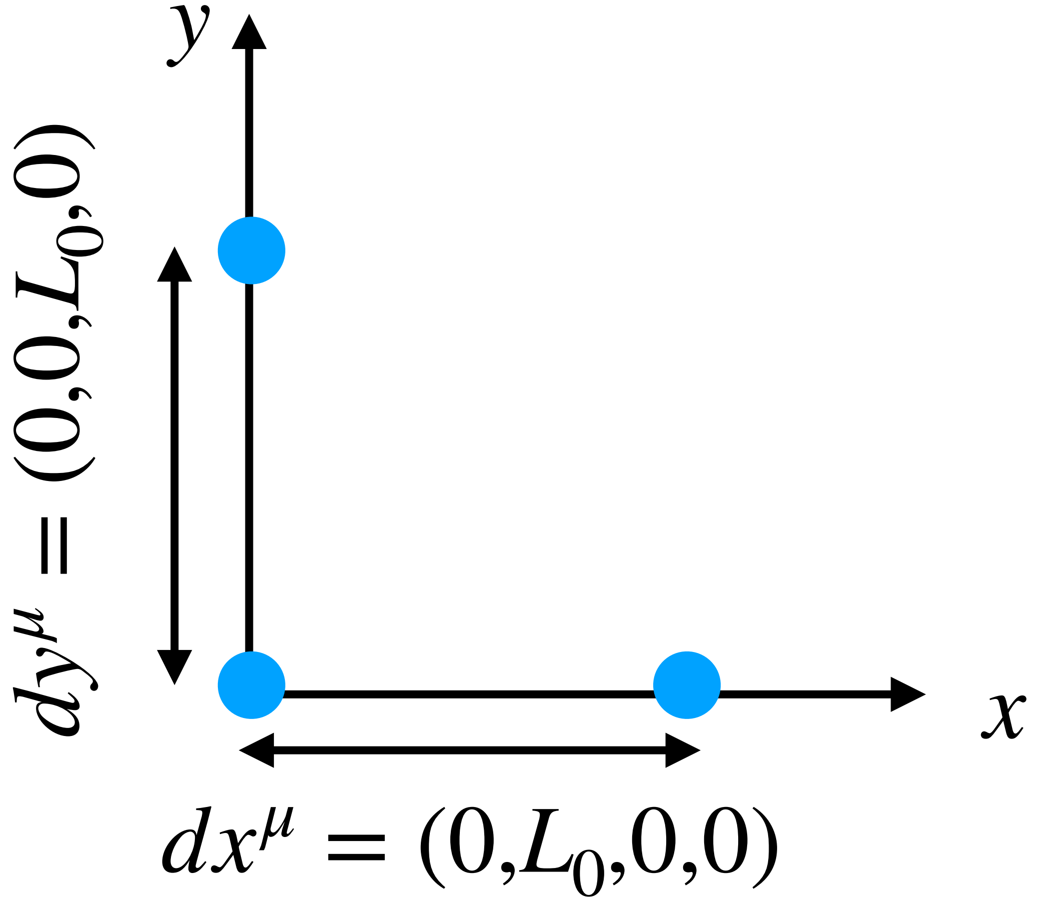

GW solutions

What do the waves actually look like?

- Proper length of dx^μ is root of the 4D spacetime interval:

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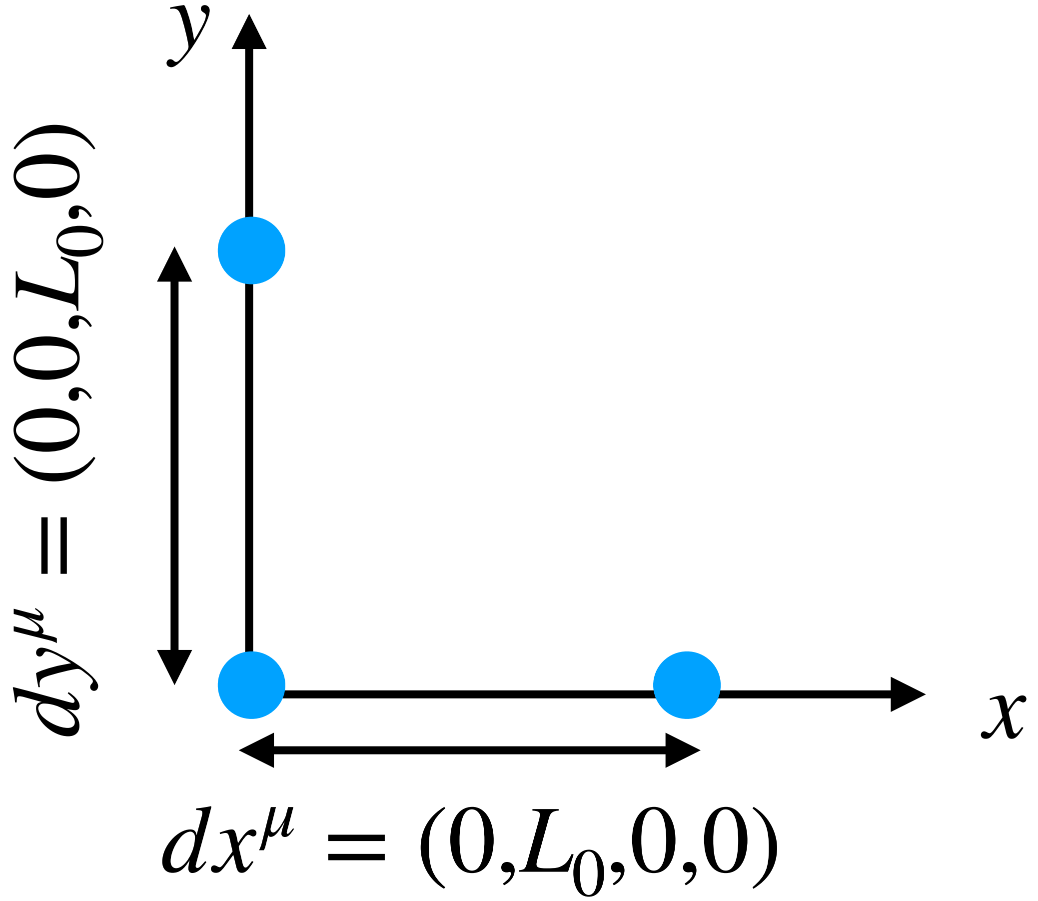

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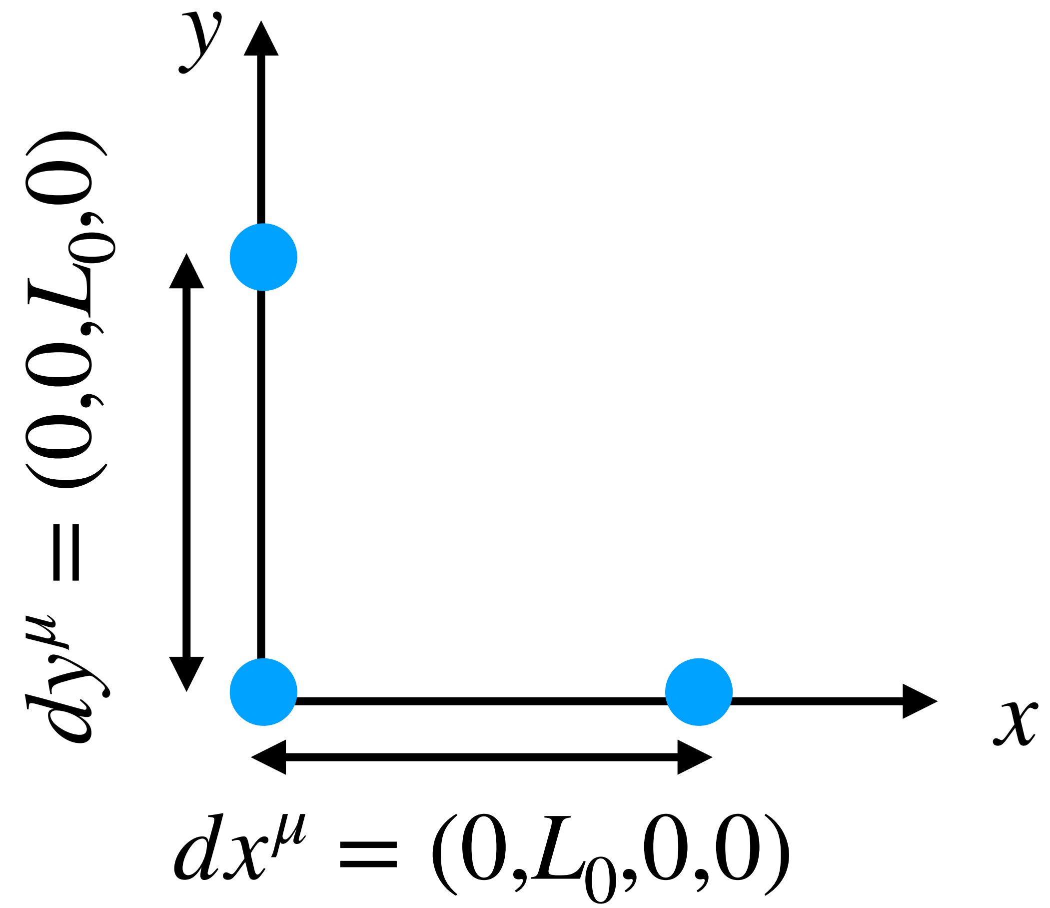

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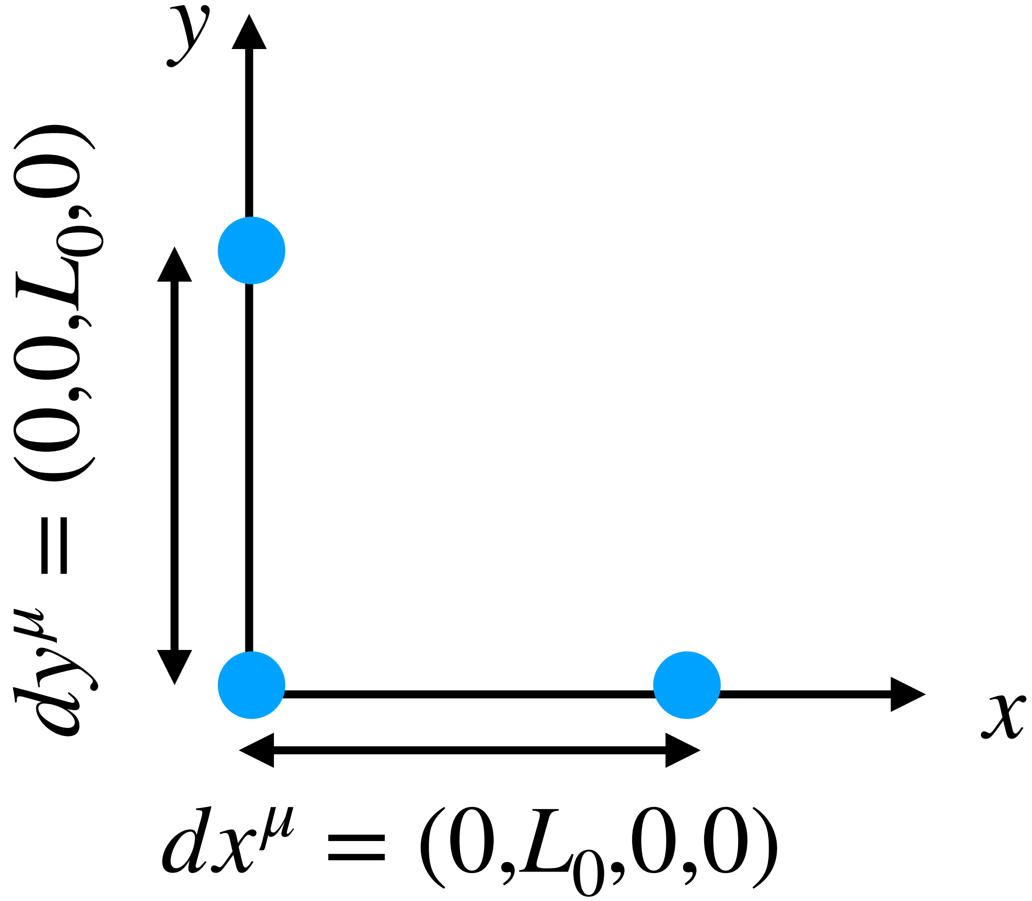

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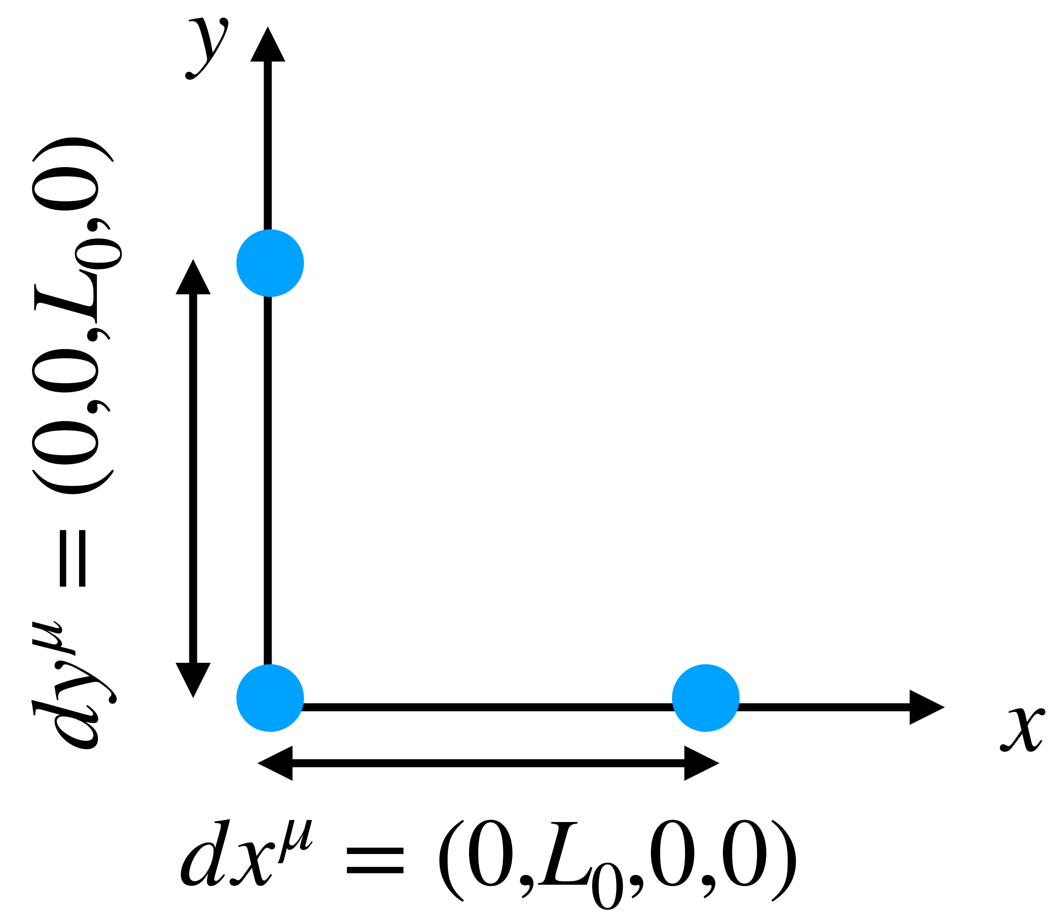

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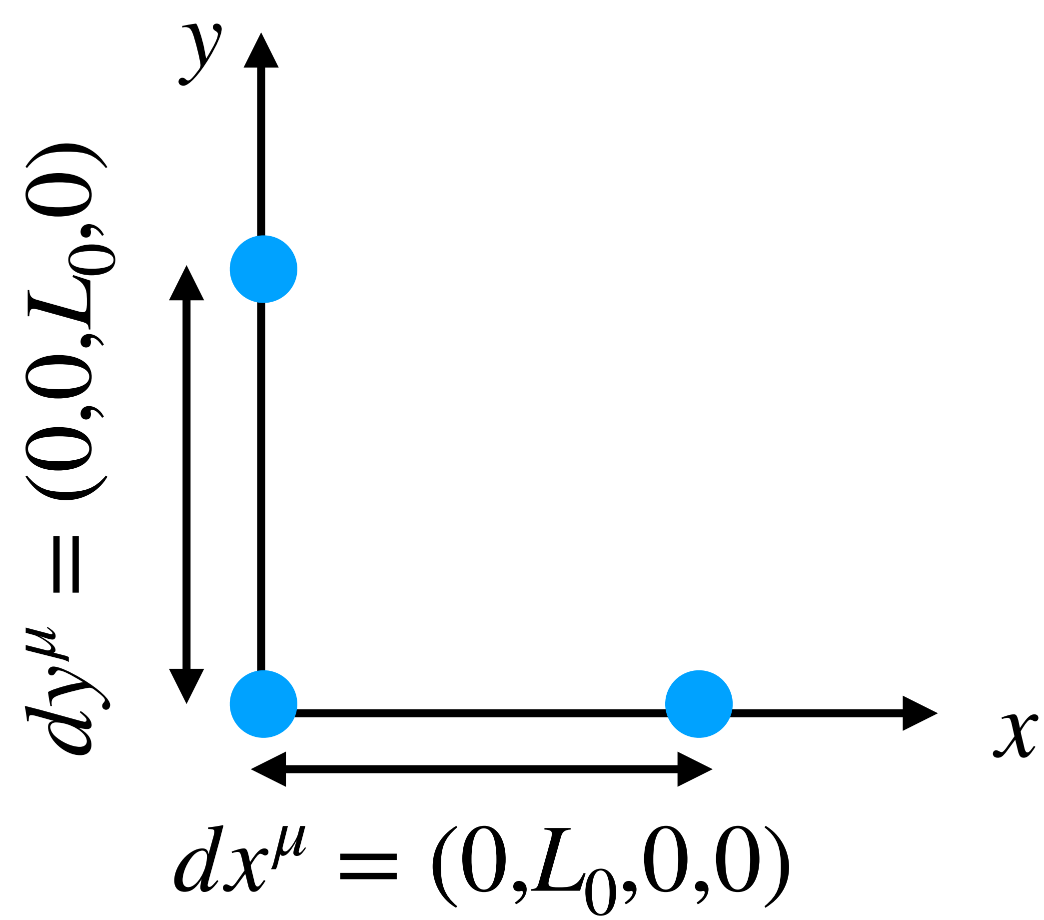

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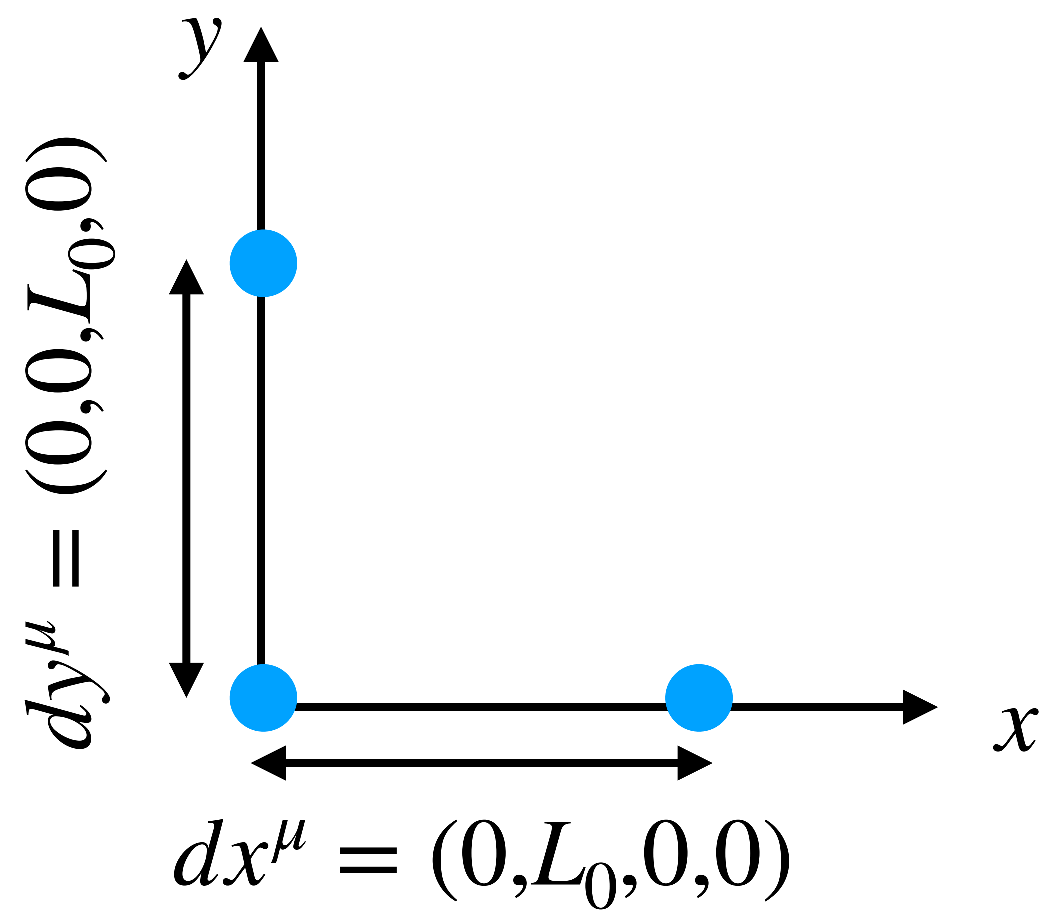
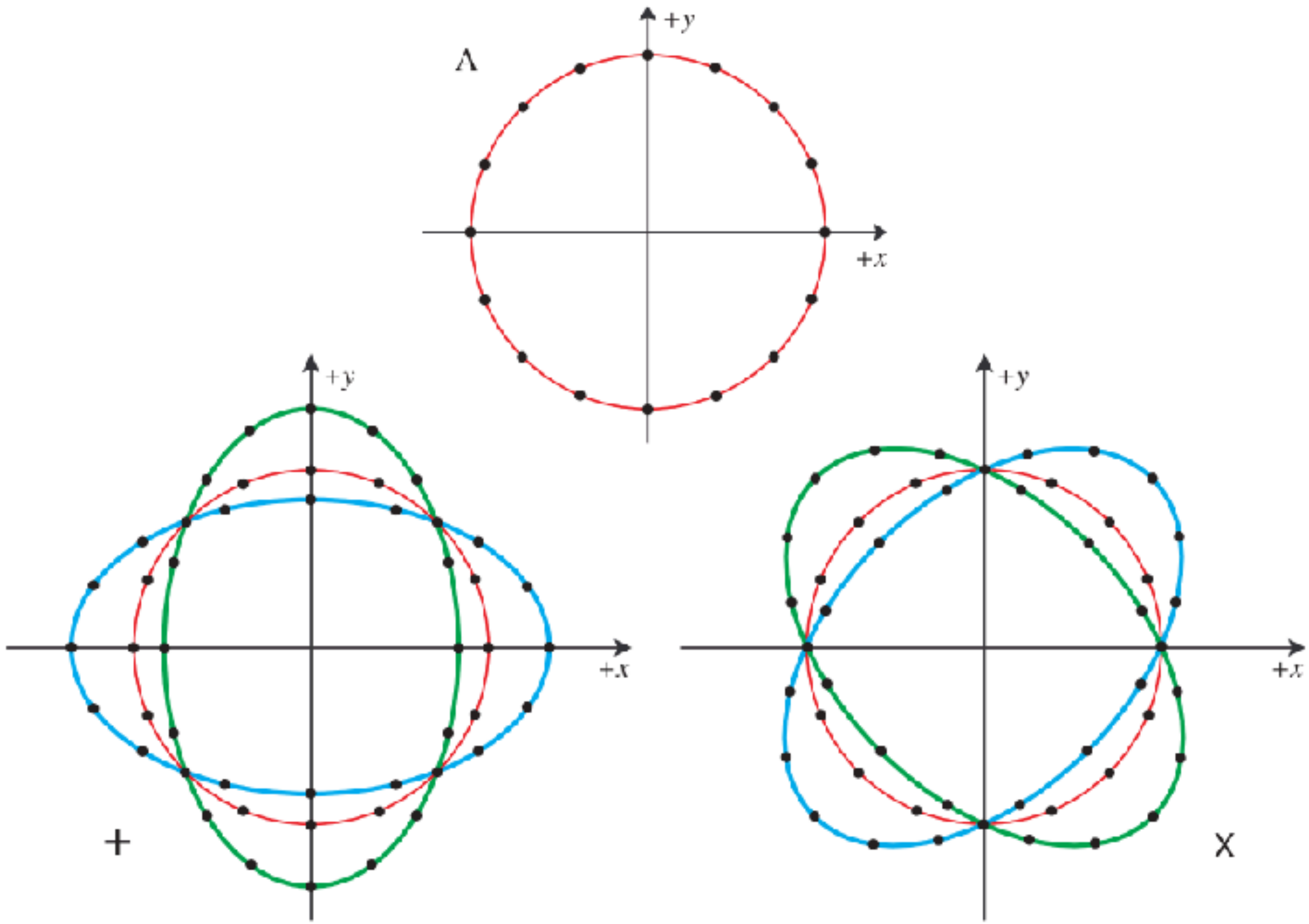
- x and y oscillations are out of phase



GW solutions

$$\left(\frac{\Delta L}{L}\right)_x \propto \cos(\omega t) \qquad \left(\frac{\Delta L}{L}\right)_y \propto -\cos(\omega t)$$

Or place test masses in a circle in the x-y plane:



GW solutions

We now want to know the amplitude of the GWs by solving the Einstein equations in the vicinity of the source (T>0 at the source):

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} \right) h_{\mu\nu} = -16\pi \frac{G}{c^4} T_{\mu\nu}$$

After a lot of maths, end up with GW for observer distance r from source:

$$\boxed{h_{ij}(r,t) = \frac{2}{r}\ \frac{G}{c^4}\ \ddot{Q}_{ij}(t - r/c)}$$

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Why 1/r?

Direct analogy to EM waves: energy carried in the wave is proportional to the amplitude squared (i.e. Poynting flux). Energy conservation => transmitted energy proportional to $1/r^2$.

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GWs carry energy away from the source. Energy carried away from source per unit time (GW luminosity) is:

$$L_{GW} = \frac{G^4}{5c^5} \langle \dddot{Q}_{ij} \dddot{Q}^{ij} \rangle$$

Averaging is over characteristic timescale (one orbital period for GWs from a BBH)

GWs from a BBH system

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$$Q_{ij} = \frac{1}{2}\mu M r_a^2 \begin{pmatrix} 1/3 + \cos(2\phi) & \sin(2\phi) & 0 \\ \sin(2\phi) & 1/3 - \cos(2\phi) & 0 \\ 0 & 0 & -2/3 \end{pmatrix}$$

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Where:

$$h_0 = 4\frac{GM}{c^4}\mu\frac{(r_a\Omega)^2}{r}$$

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$M = 20M_\odot;\ \mu = 1/4;\ r = 40\ \mathrm{Mpc};\ r_a = 6\ r_g$

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$$\Longrightarrow \; h_0 \approx 4 \times 10^{-21}$$

TINY!

GWs from a BBH system

What about the GW luminosity?

$$L_{GW} = \frac{G^4}{5c^5}\langle \dddot{Q}_{ij}\dddot{Q}^{ij}\rangle = \frac{G^4}{5c^5}\frac{(M_1M_2)^2M}{r_a^5}$$

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$M_1 = M_2 \implies$

$$L_{GW} = \frac{2}{5}\frac{c^5}{G}\left(\frac{r_g}{r_a}\right)^5$$

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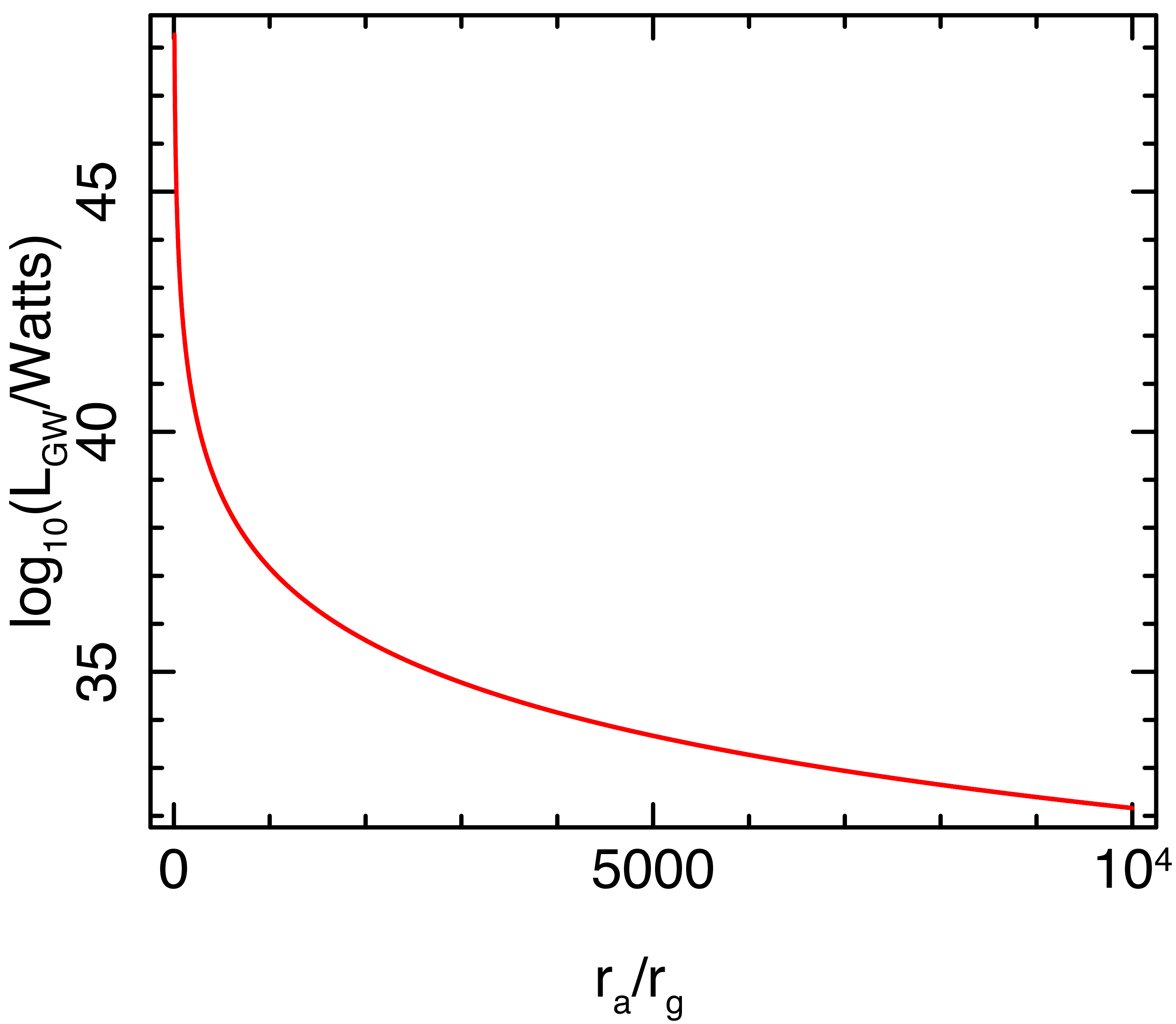

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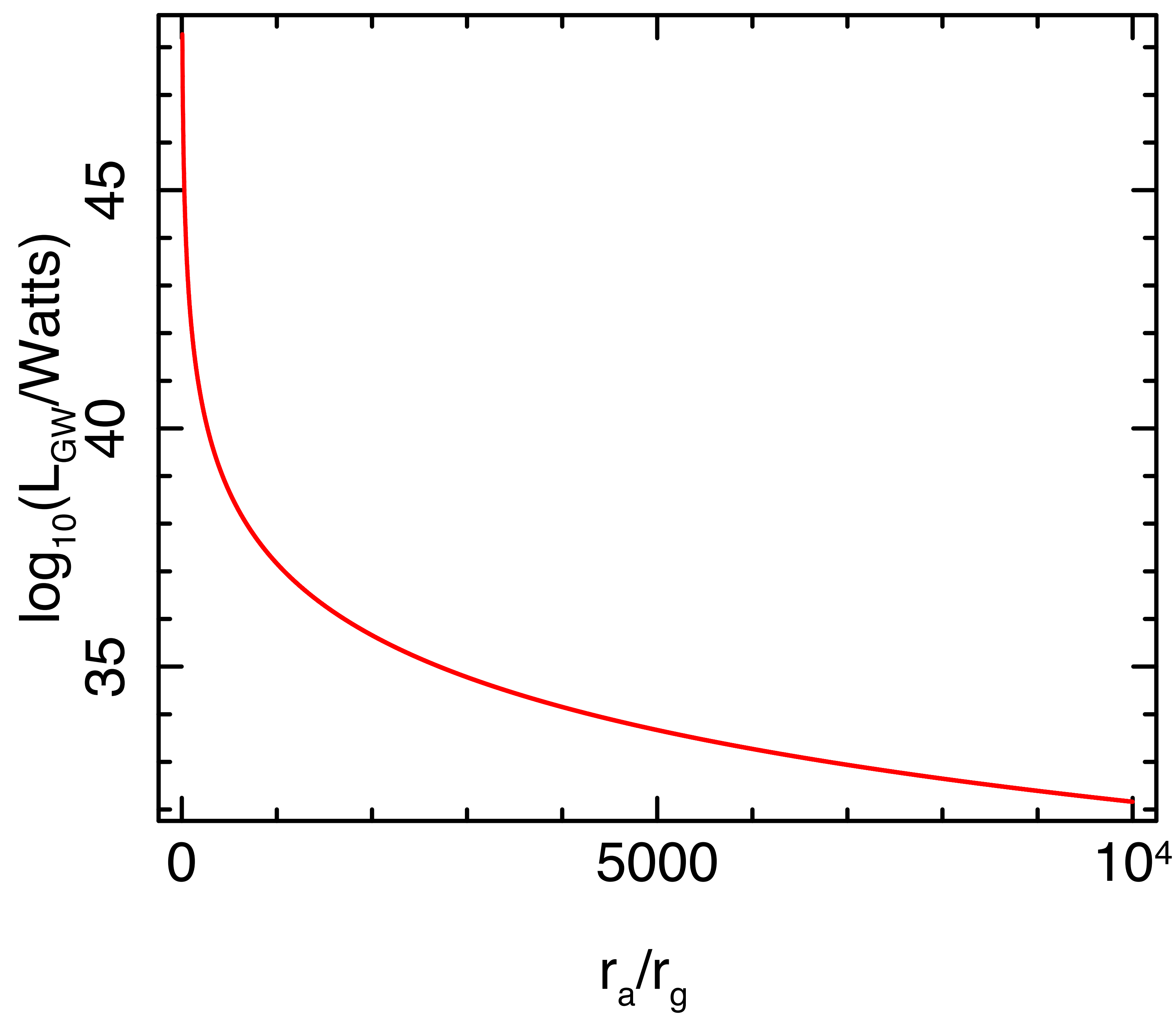

HUGE! Why?

- Amplitude reduces with distance, this is luminosity lost by the system.
- Bending spacetime takes **a lot** of energy!

Binary evolution

GWs take energy out of the system, therefore binary orbit (and eccentricity) shrinks!

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$$\frac{dr_a}{dt} = \frac{dr_a}{dE}\frac{dE}{dt} = \frac{dr_a}{dE}L_{GW}$$

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Rate of change of orbital period P:

$$\dot{P} \equiv \frac{dP}{dt} = \frac{dP}{dr_a}\frac{dr_a}{dt} = \frac{3}{2}\frac{2\pi}{(GM)^{1/2}}r_a^{1/2}\frac{dr_a}{dt}$$

Kepler's law

$$P^2 = \frac{(2\pi)^2 r_a^3}{GM}$$

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Eliminate r_a using Kepler's law:

$$\dot{P} = -\frac{96}{5}(2\pi)^{8/3}\left(\frac{G\mathcal{M}}{c^3}\right)^{5/3}P^{-5/3}$$

Kepler's law

$$P^2 = \frac{(2\pi)^2 r_a^3}{GM}$$

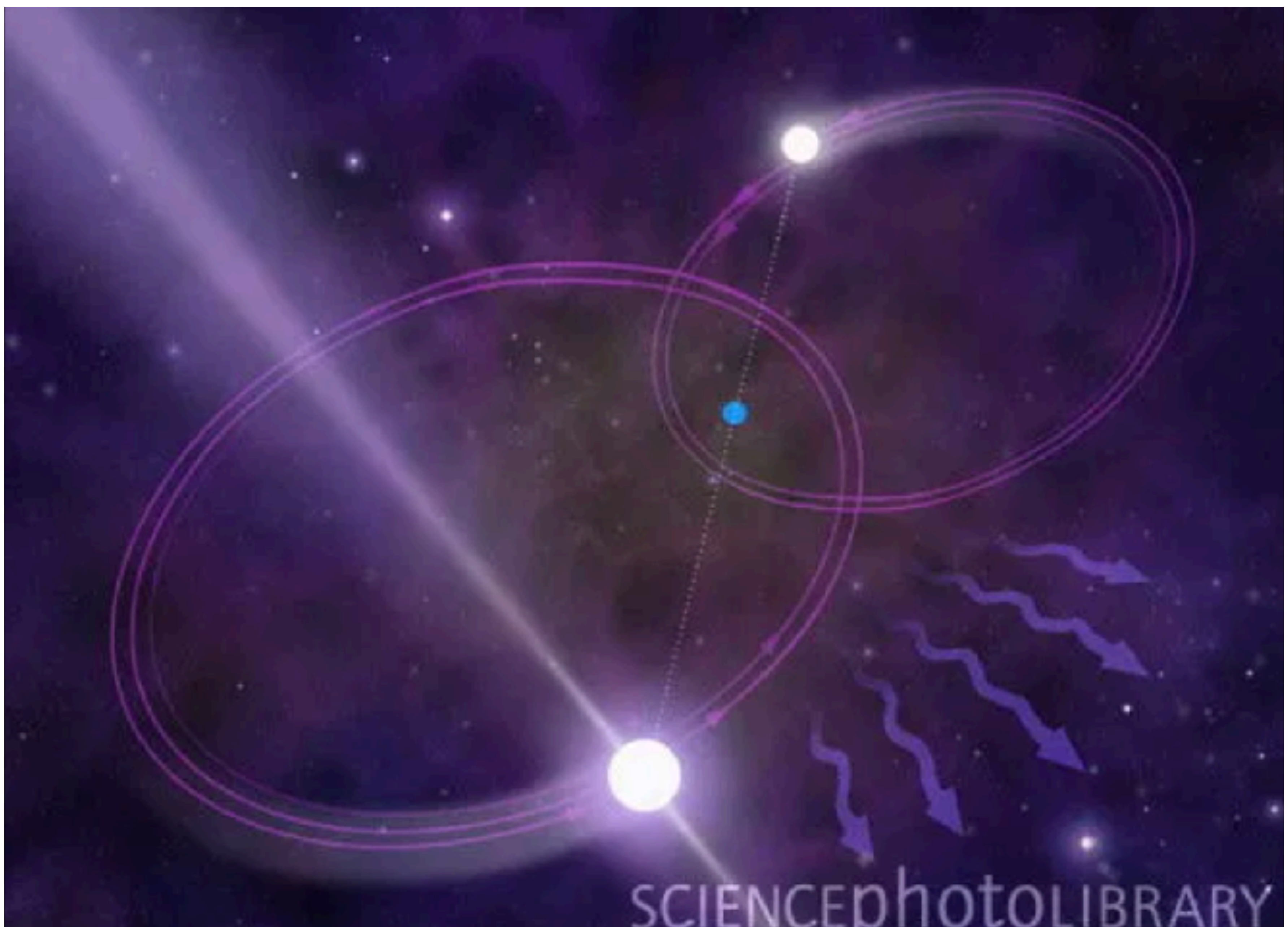
Chirp mass

$$\mathcal{M} = \left(\frac{M_1^3M_2^3}{M}\right)^{1/5}$$

Binary evolution

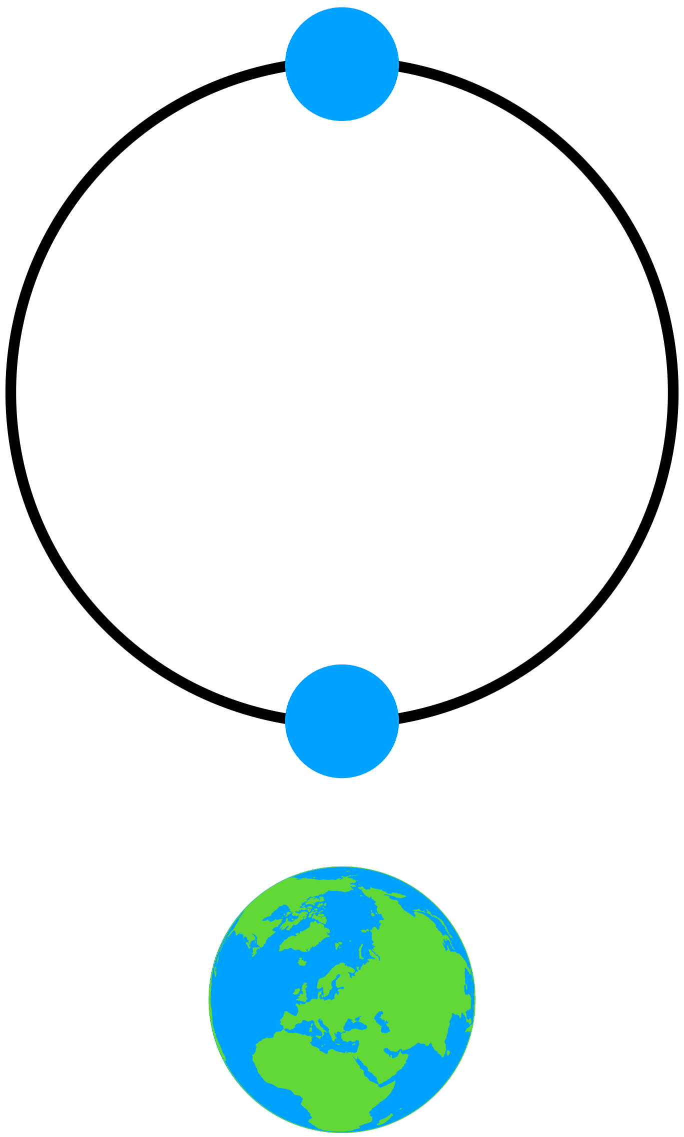
Hulse-Taylor binary:

- Binary neutron star system, one of the NSs is a pulsar, discovered in 1974.
- Doppler shifts cause small variations in pulse period that can be used to accurately measure both NS masses (and orbital period).
- Therefore know **exactly** how the orbital period should evolve due to GWs.

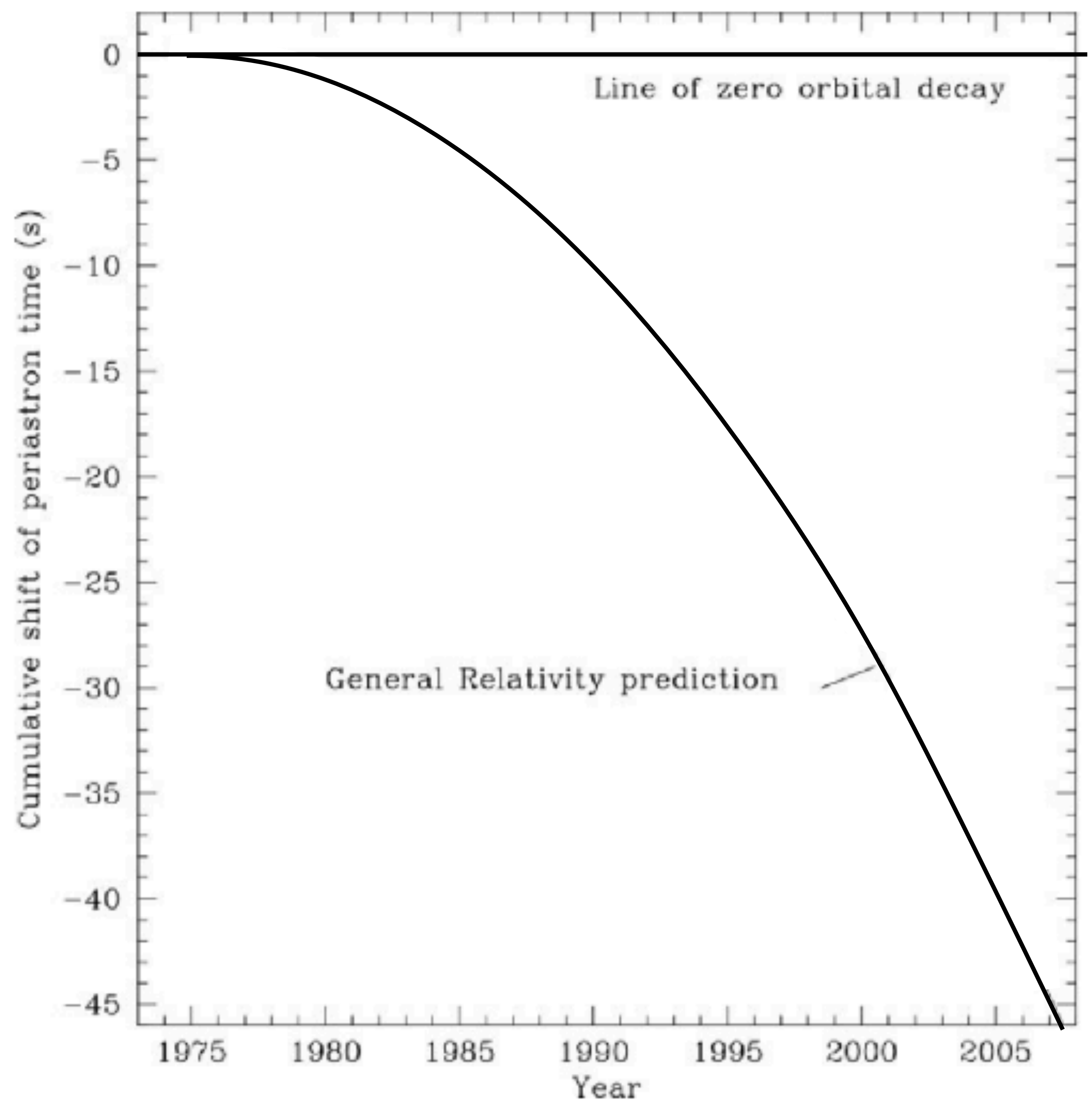


Binary evolution

- Periastron should come earlier and earlier each orbit compared with if the orbital period were constant.
- Can therefore measure the build up of orbital decay over many orbits.



$$\dot{P} = -\frac{96}{5}(2\pi)^{8/3}\left(\frac{G\mathscr{M}}{c^3}\right)^{5/3} P^{-5/3}$$



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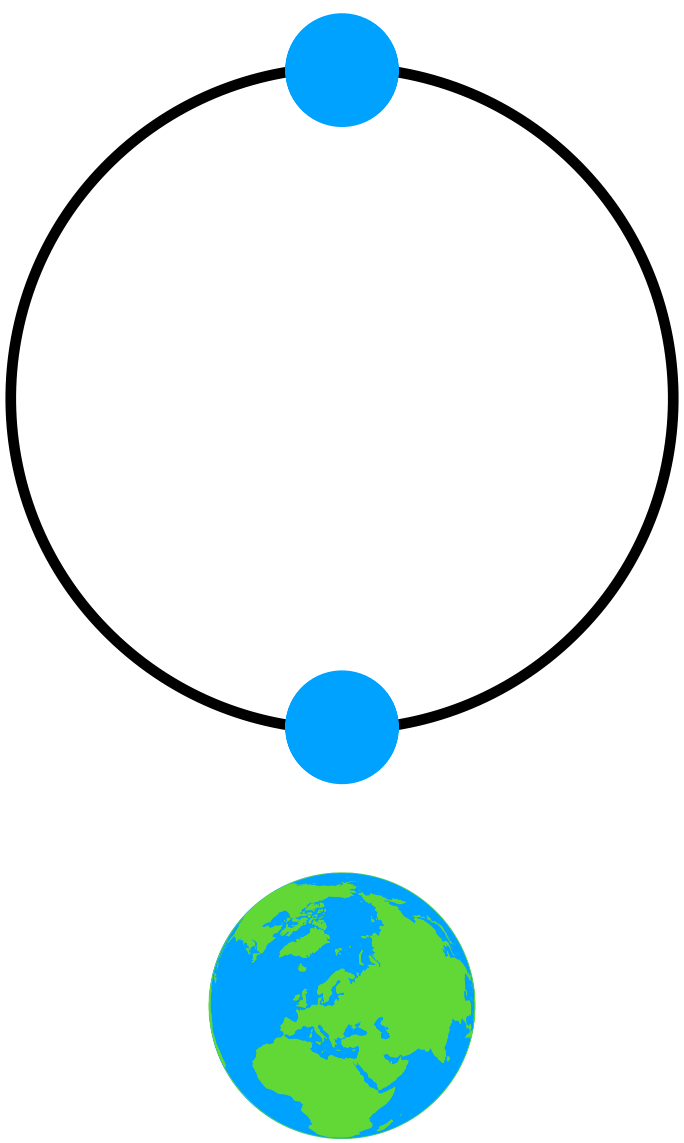

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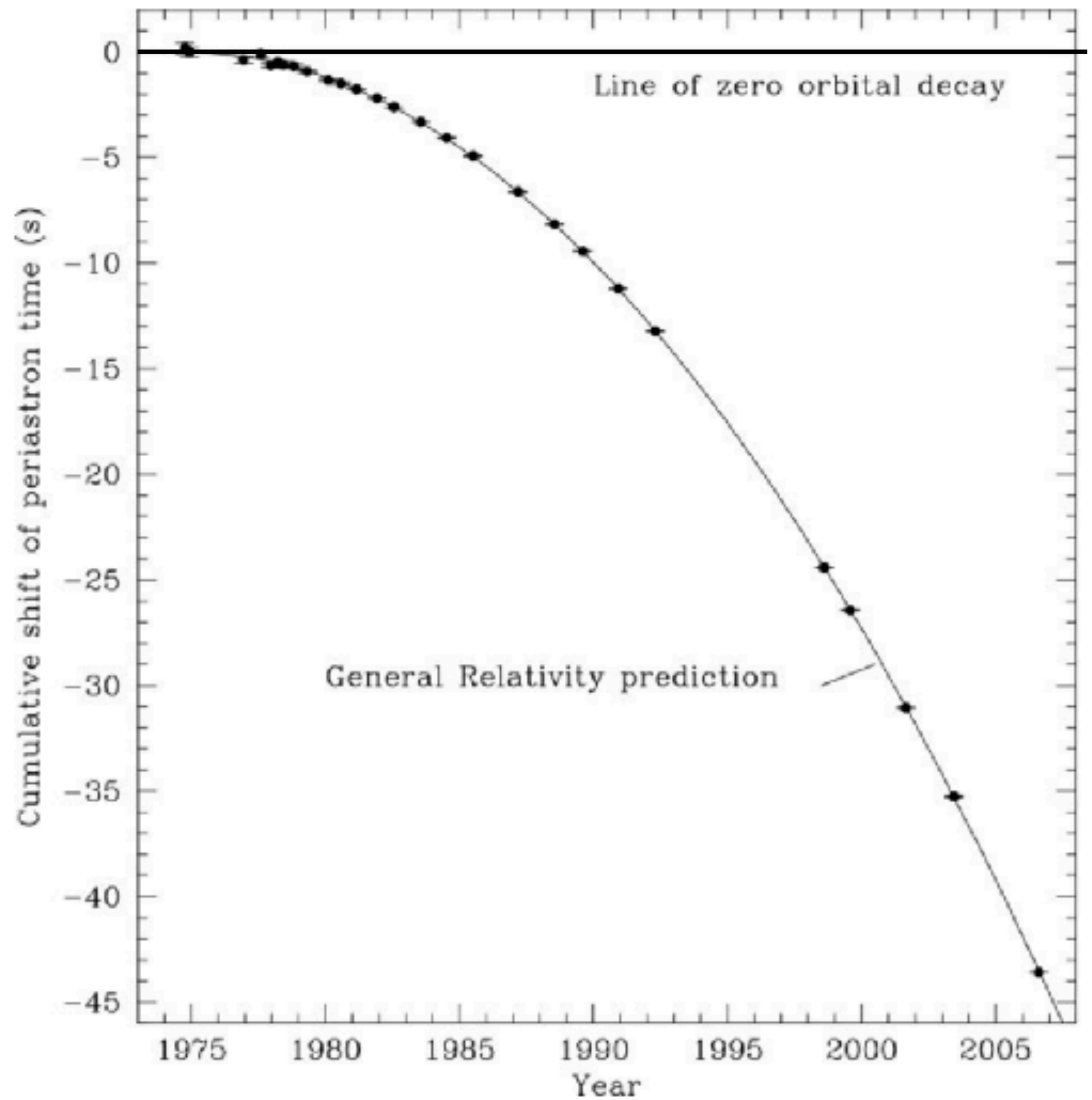

Binary evolution

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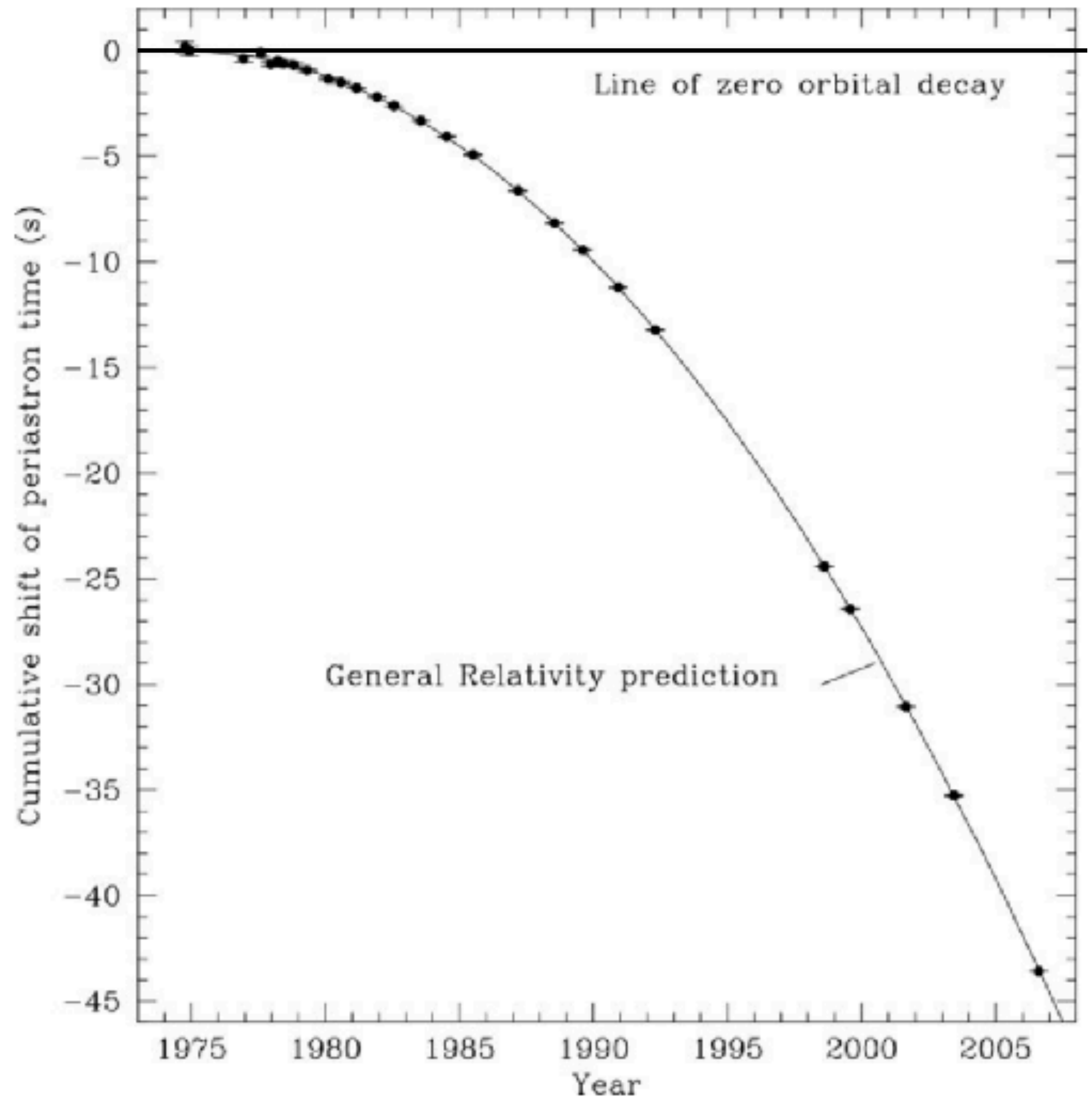
- Can therefore measure the build up of orbital decay over many orbits.

The Nobel Prize in Physics 1993

Photo from the Nobel Foundation archive.
Russell A. Hulse
Prize share: 1/2

Photo from the Nobel Foundation archive.
Joseph H. Taylor Jr.
Prize share: 1/2



GW waveform

- GW frequency: $f = 2/P$

$$\dot{f} = \frac{df}{dP}\dot{P}$$

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$$f_0 = f(t=0)$$

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GW waveform

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- Waveform: $h(t) = h_0(t)\ \cos\varphi(t)$

- Phase: $\varphi(t) = \varphi_0 + \displaystyle\int_{t'=0}^{t'=t} 2\pi f(t')t'dt' = 2\pi\left(ft + \frac{1}{2}\dot{f}t^2\right) + \varphi_0$

GW waveform

$$h(t) = h_0(t)\ \cos(2\pi f t + \pi \dot{f} t^2)$$

Numerical Relativity

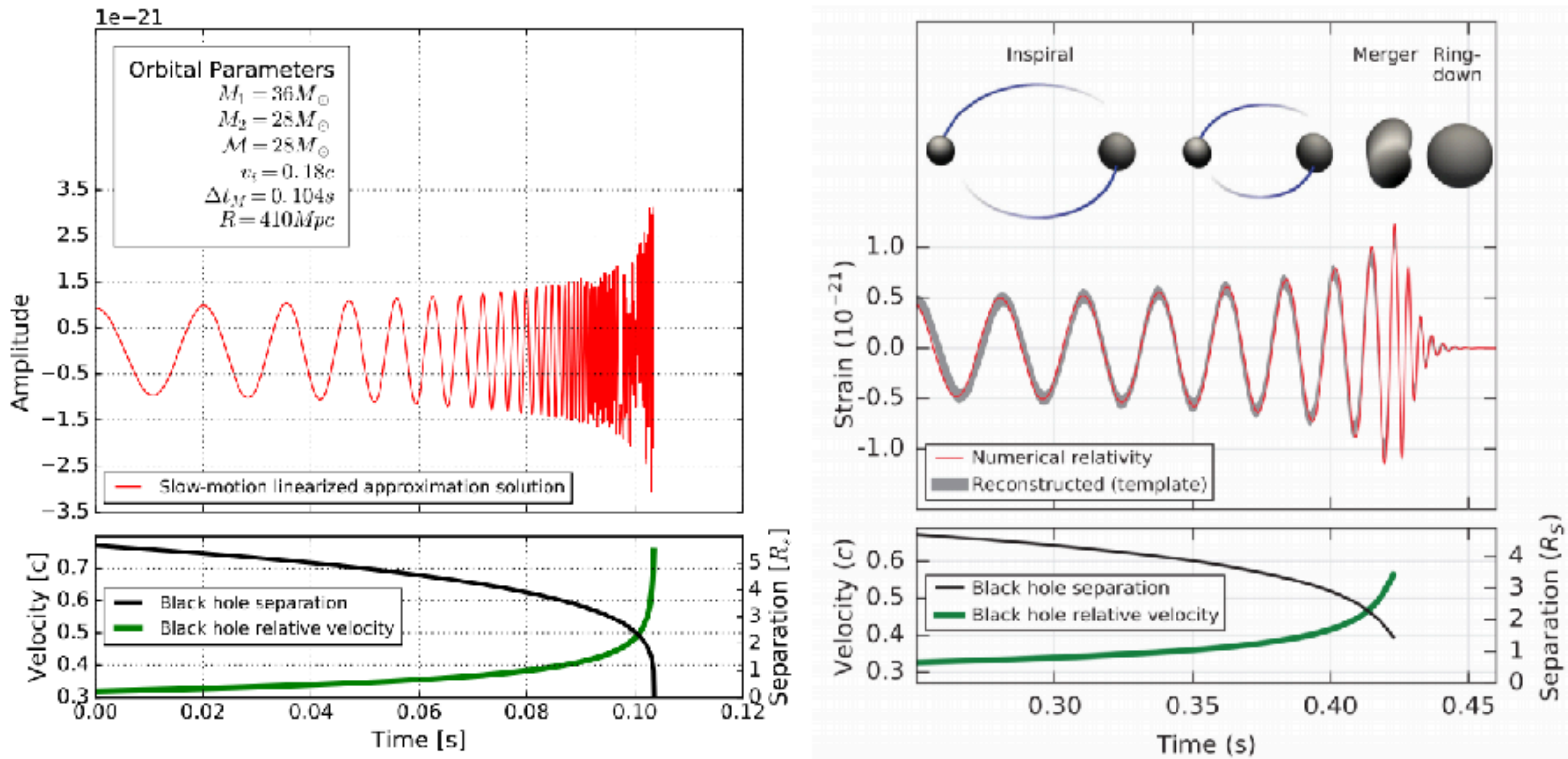
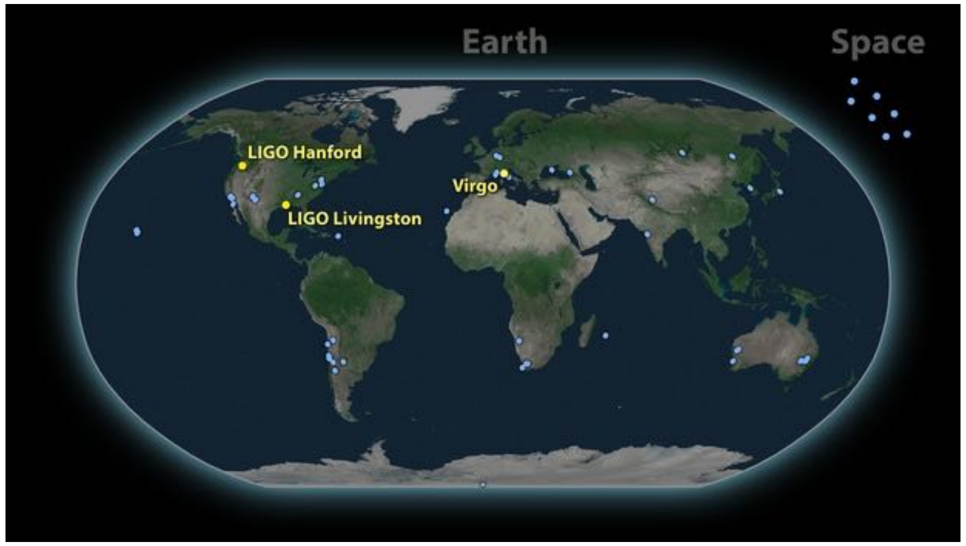


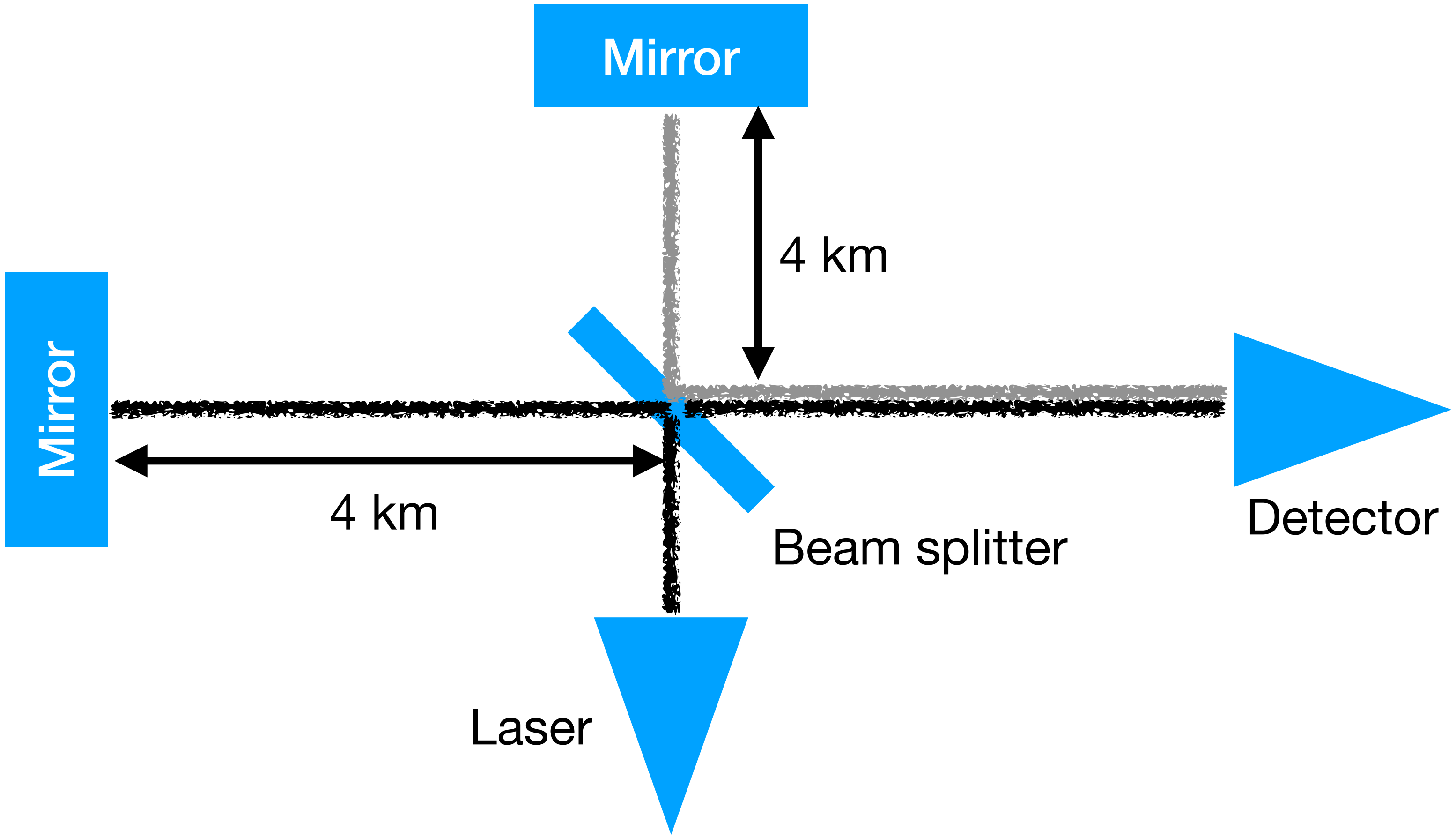
Figure 5.3: (Left) Plot of the binary black hole waveform as a function of time, as well as the black hole separation and relative velocity, all as calculated from our slow-motion approximation to the linearized theory of general relativity. (Right) Computer simulations solving Einstein's equations numerically for the waveform of coalescing binary black holes [9]. Note the strength of the velocities, especially right before the merger takes place (as well as the shape of the waveform) - here it becomes clear that our approximation is beginning to break down.

LIGO/Virgo

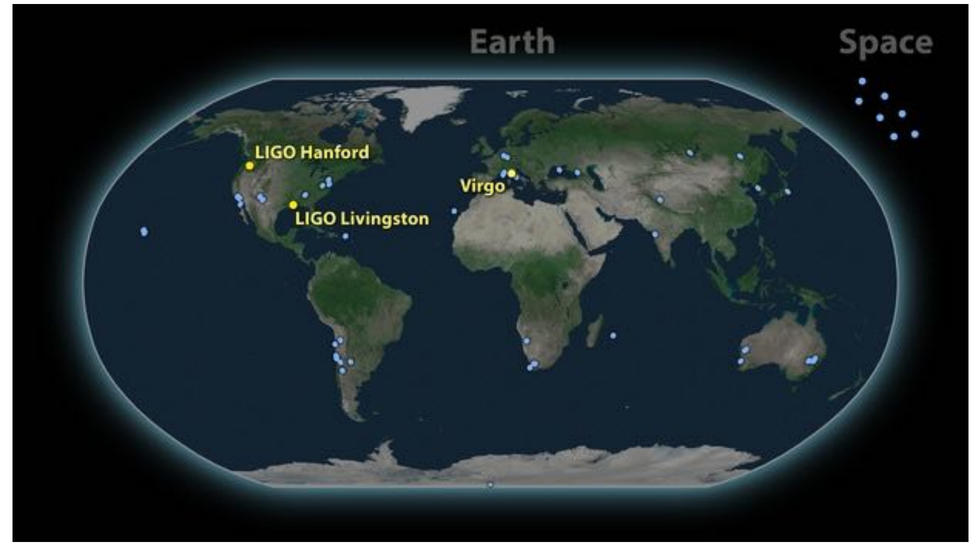
LIGO/Virgo



Michelson interferometer



LIGO/Virgo



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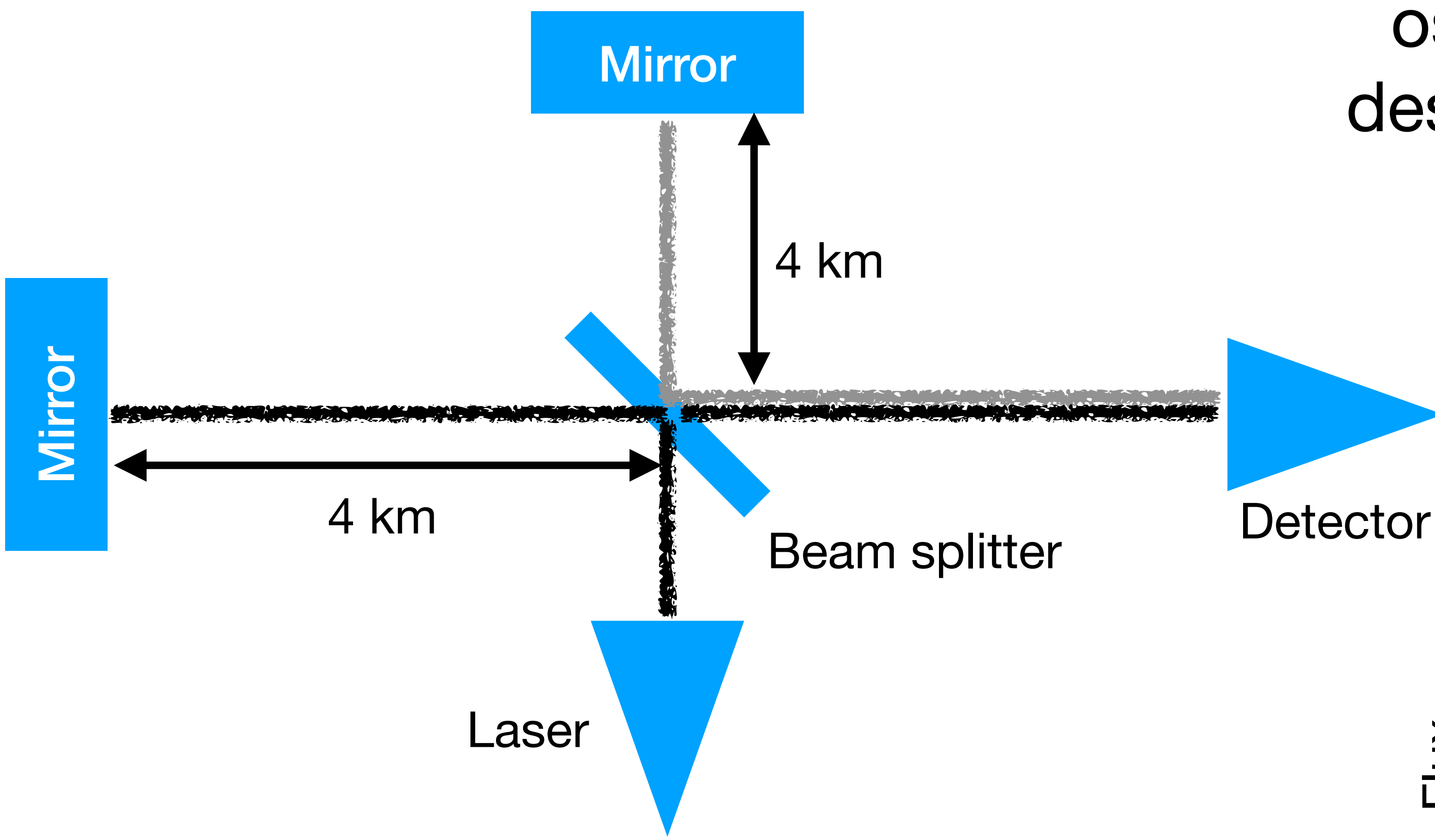
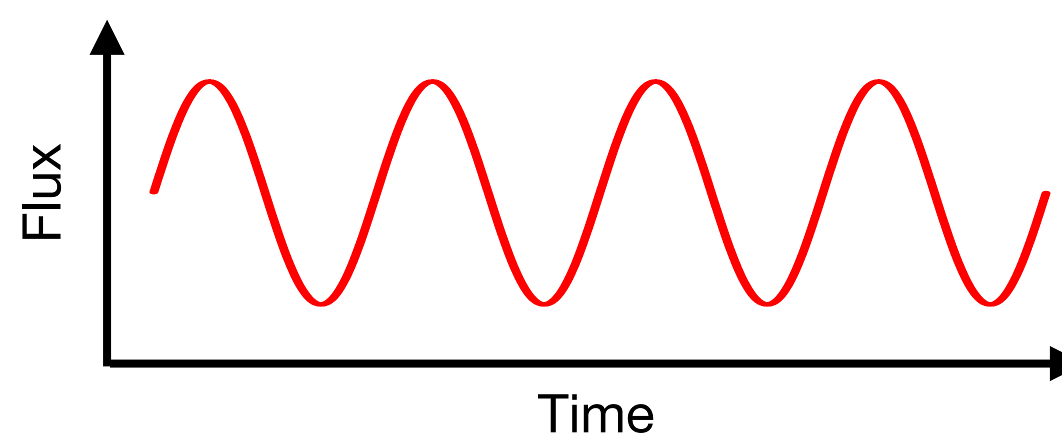


As mirrors move, detected flux oscillates due to constructive and destructive interference between the two beams.



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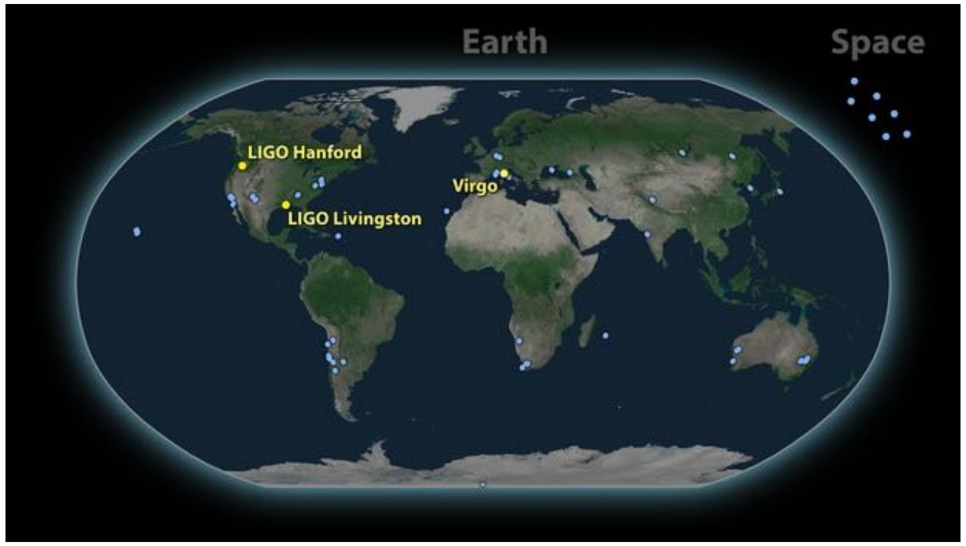

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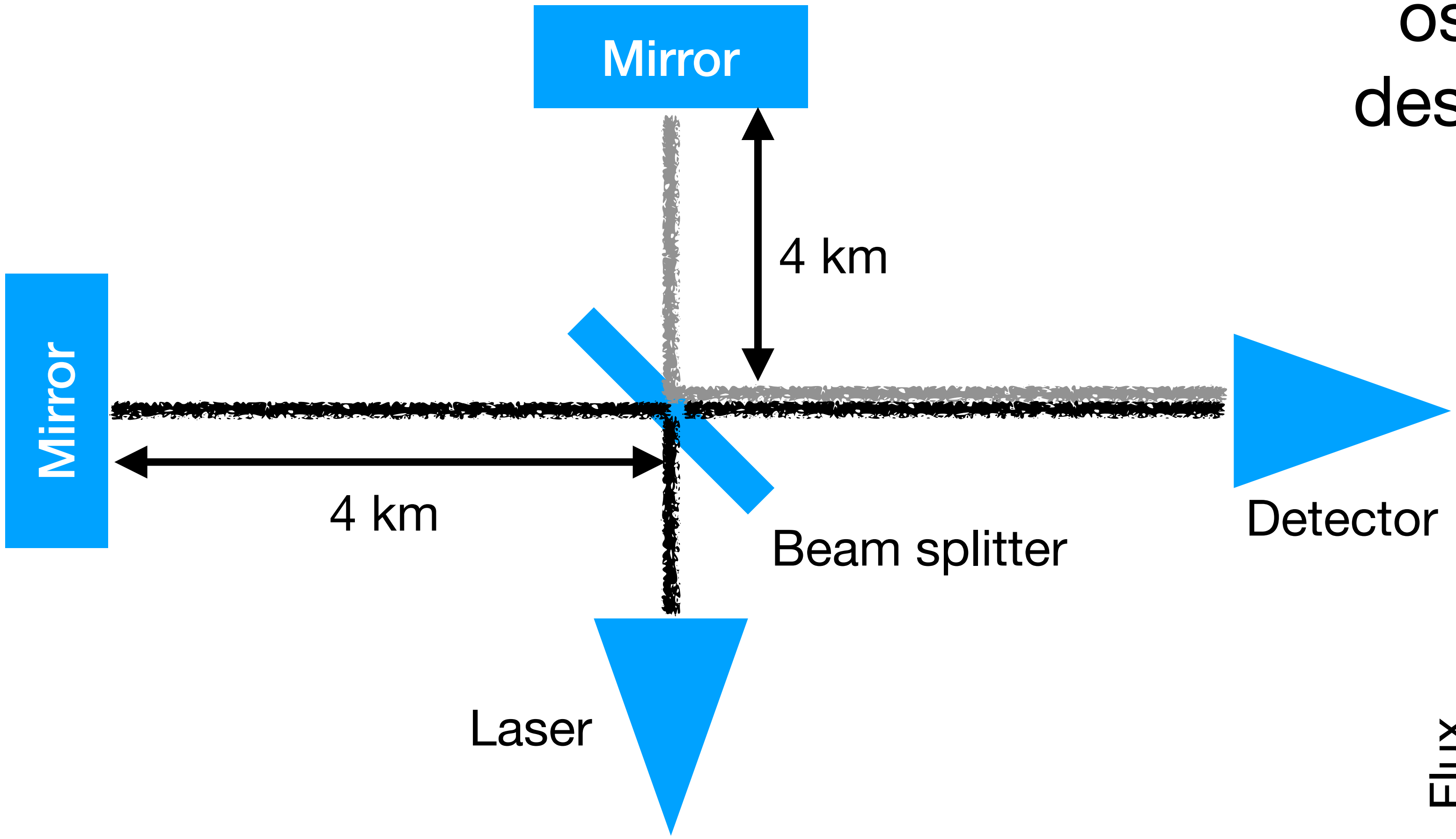

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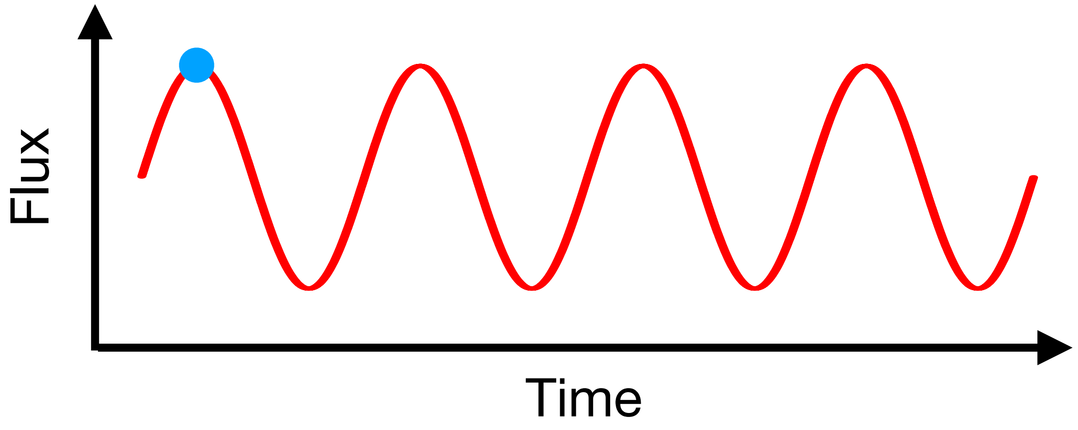
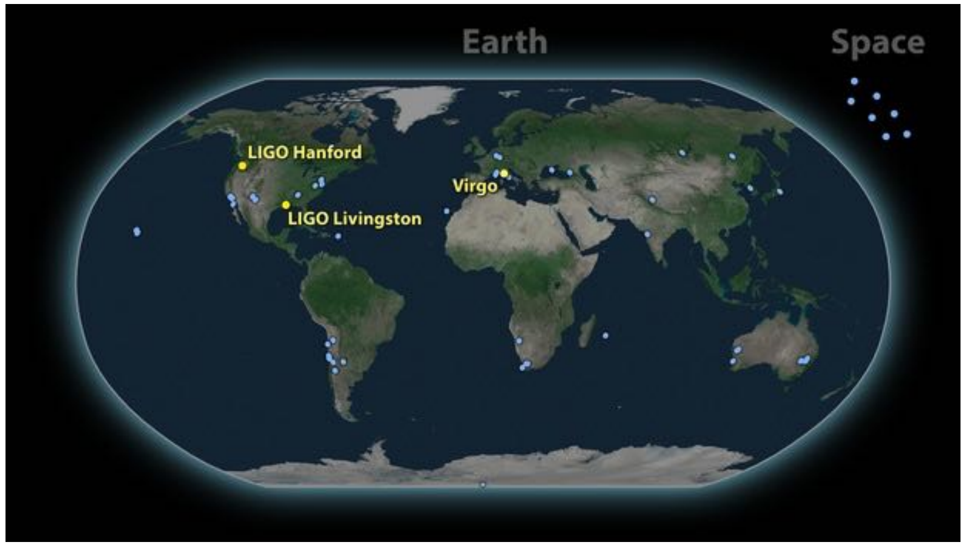


LIGO/Virgo



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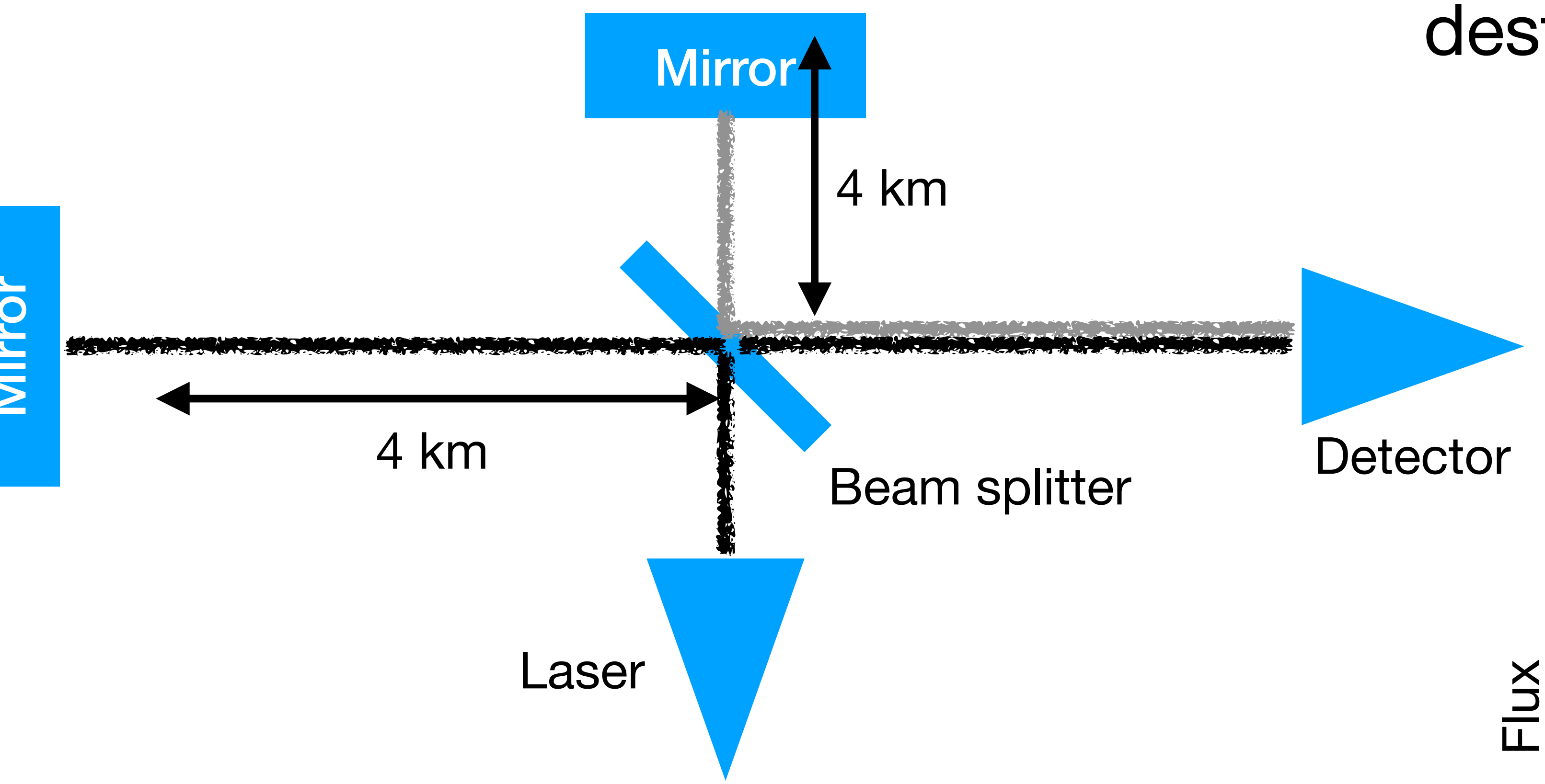

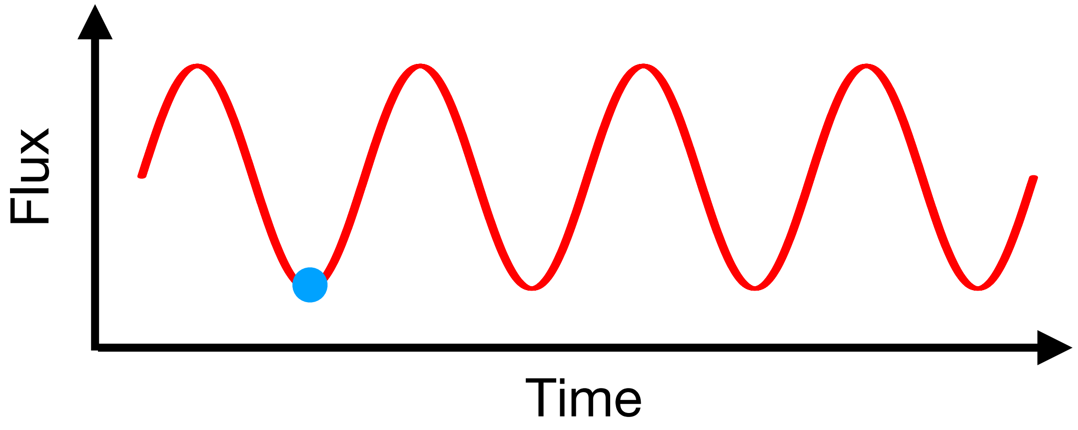

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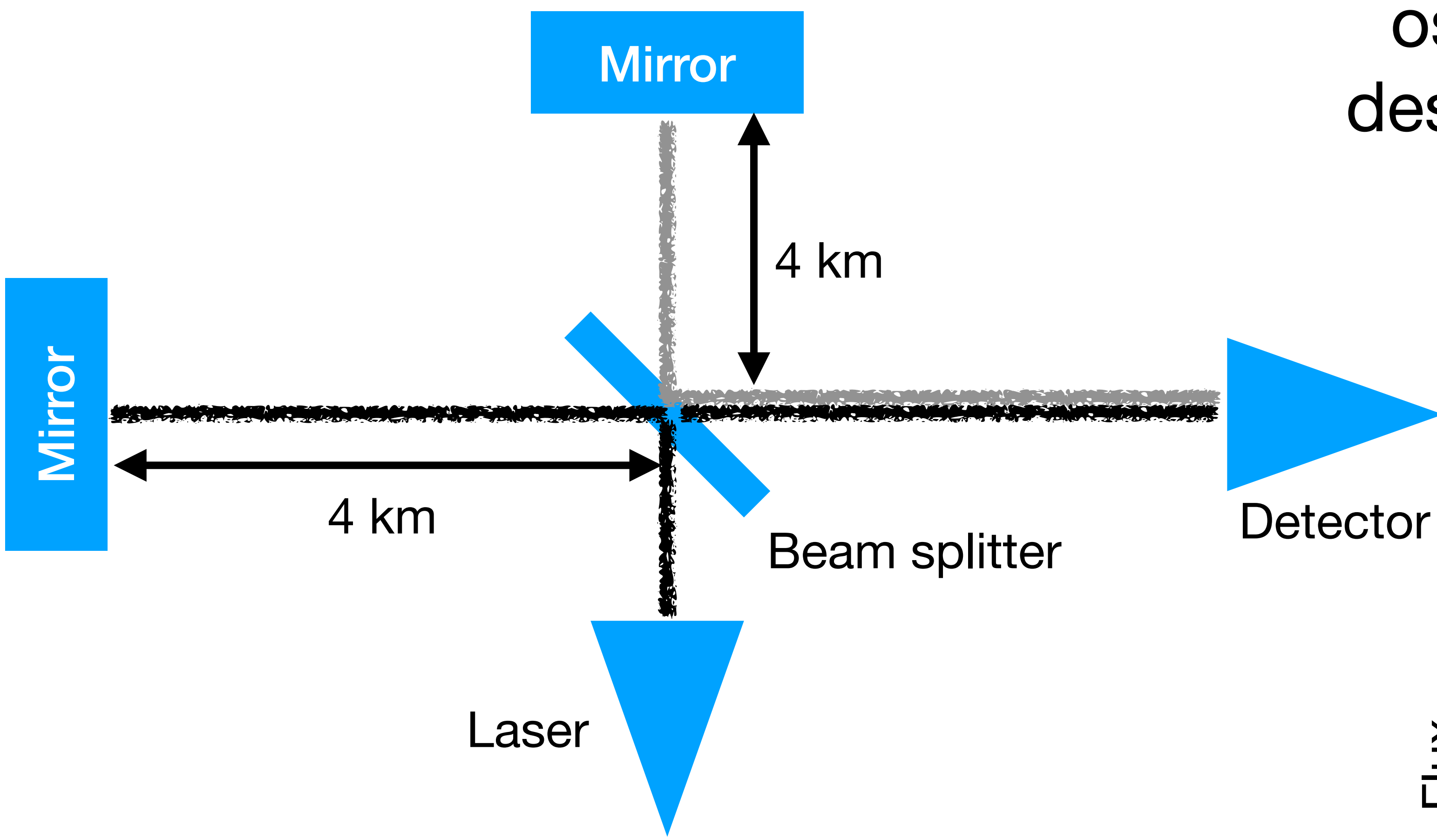
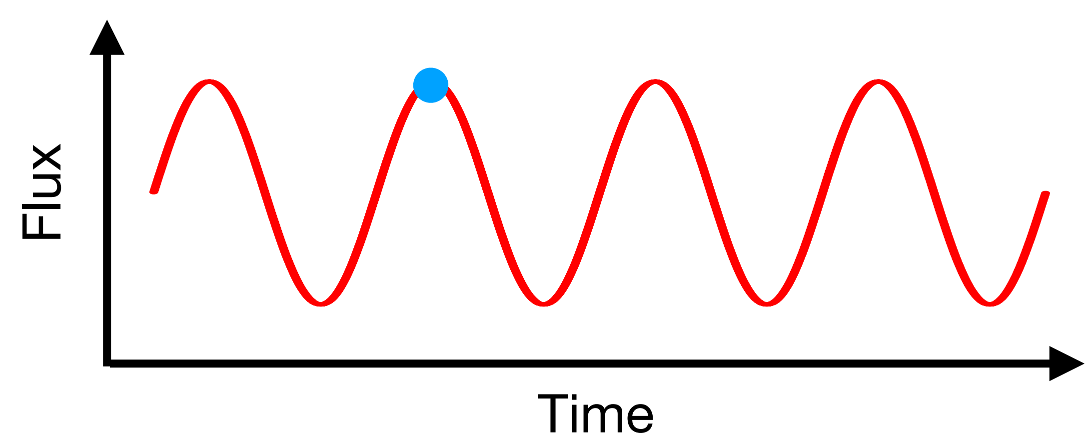
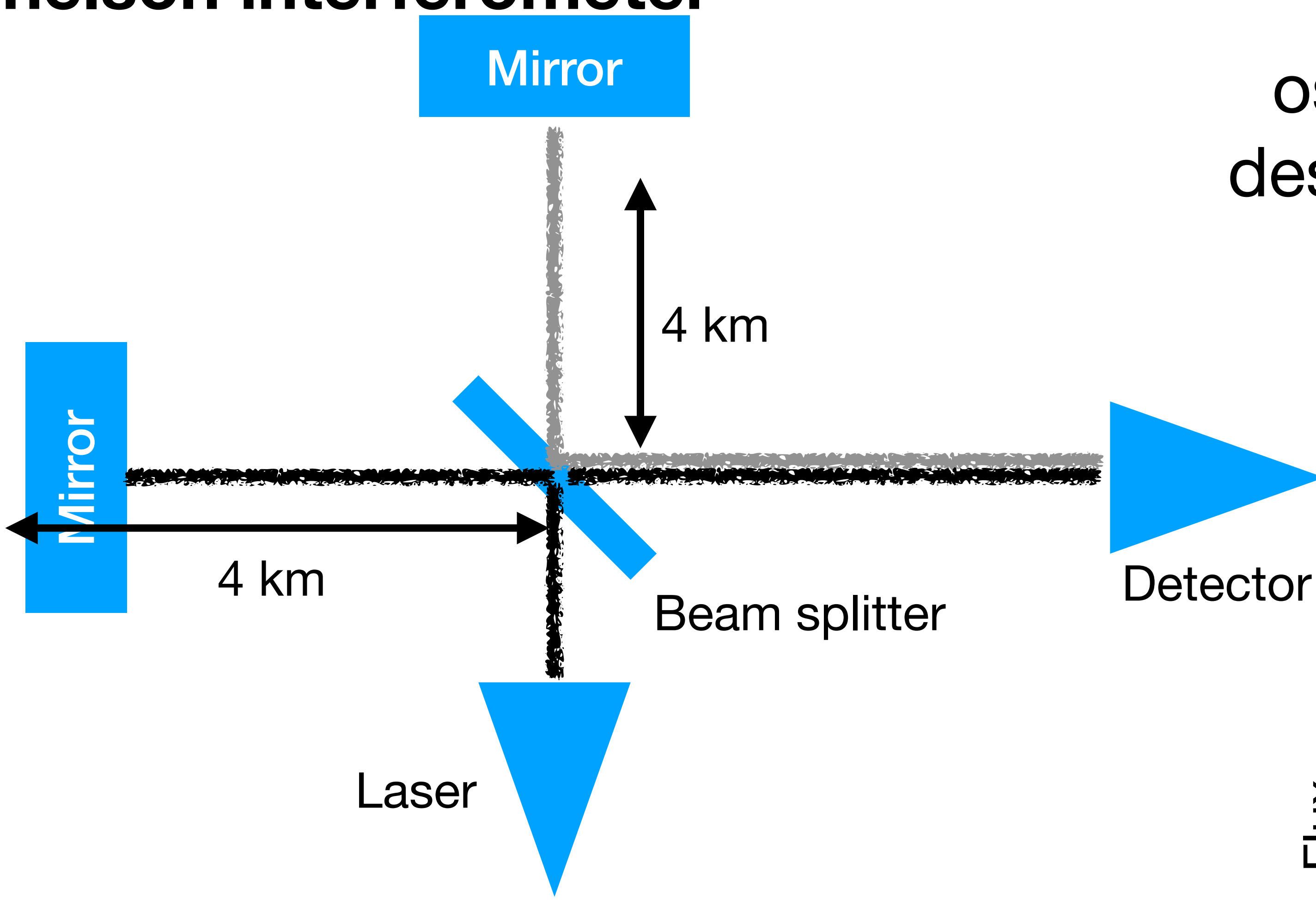




LIGO/Virgo

Michelson interferometer



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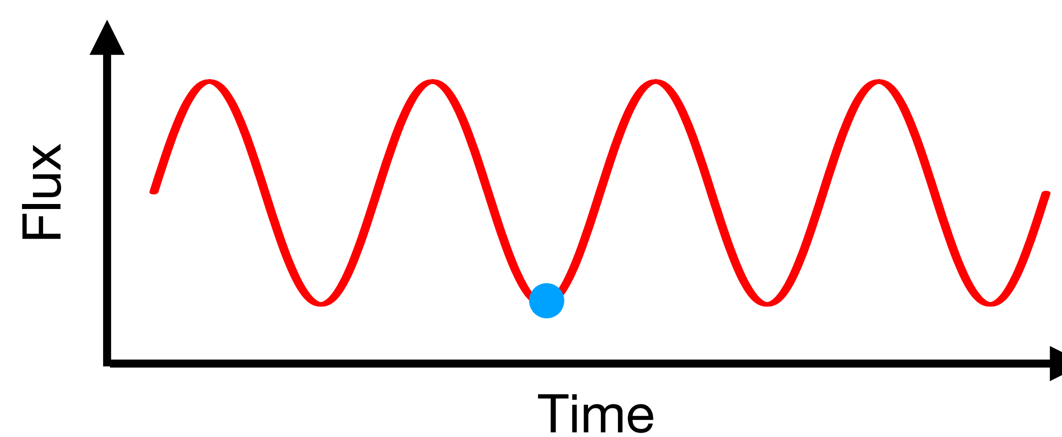
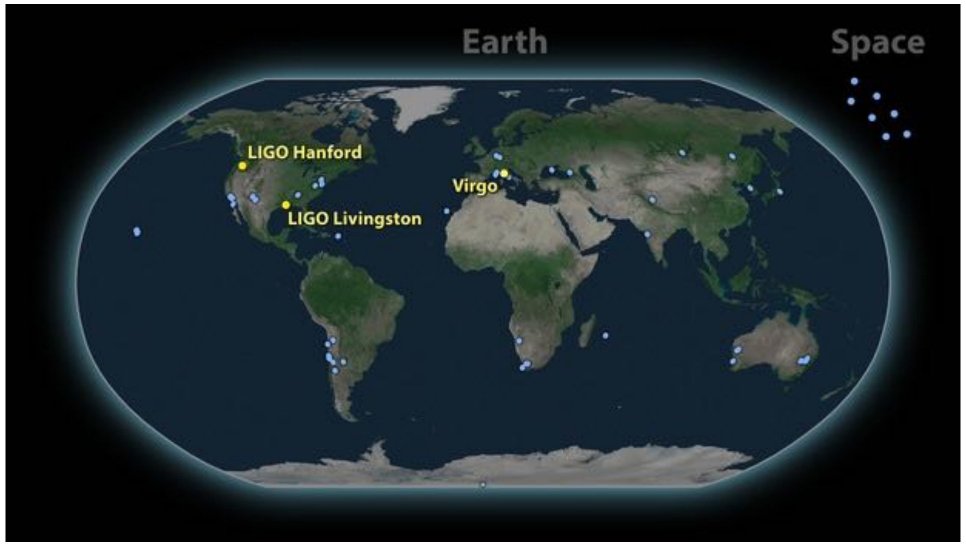


LIGO/Virgo



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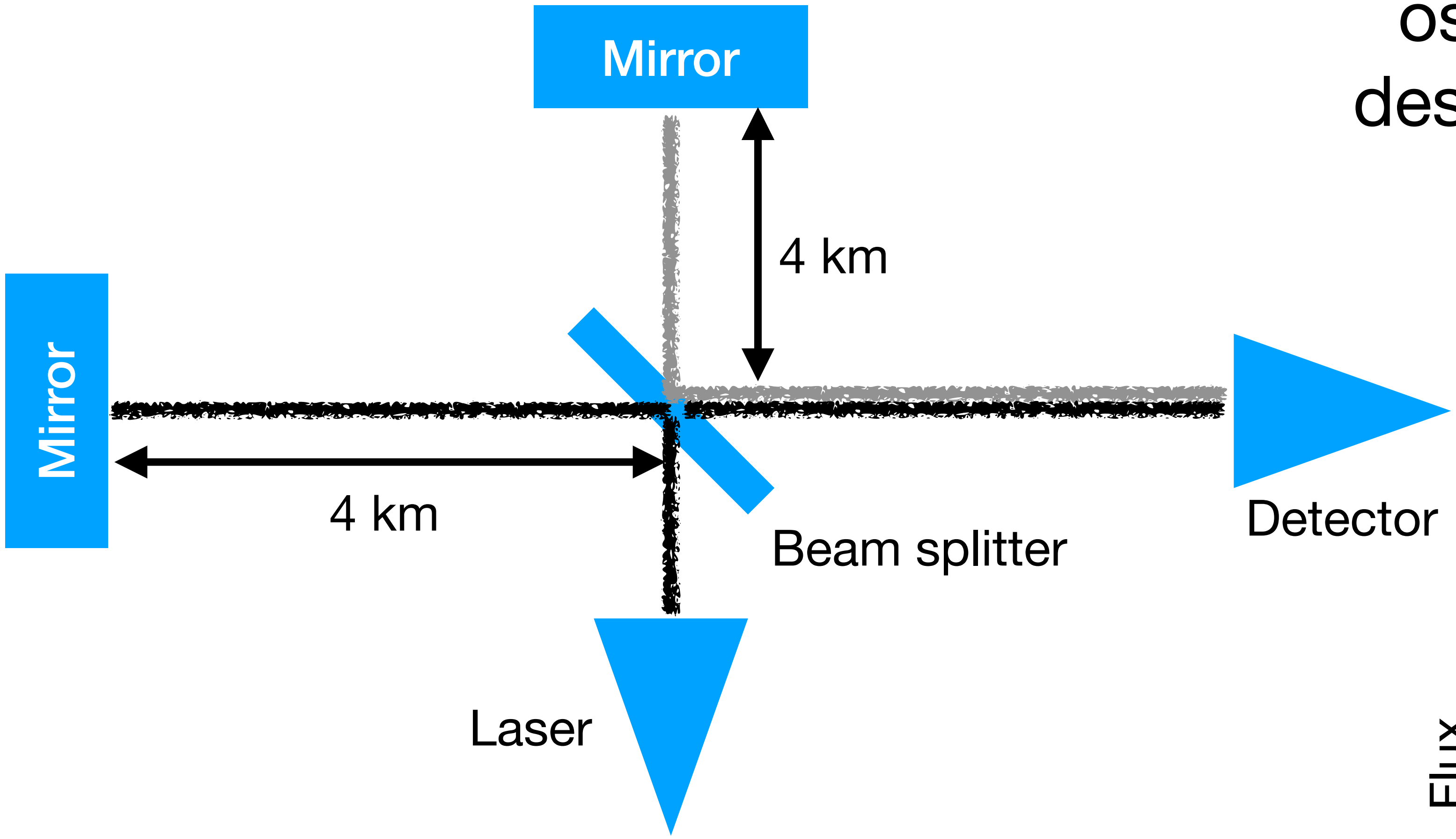

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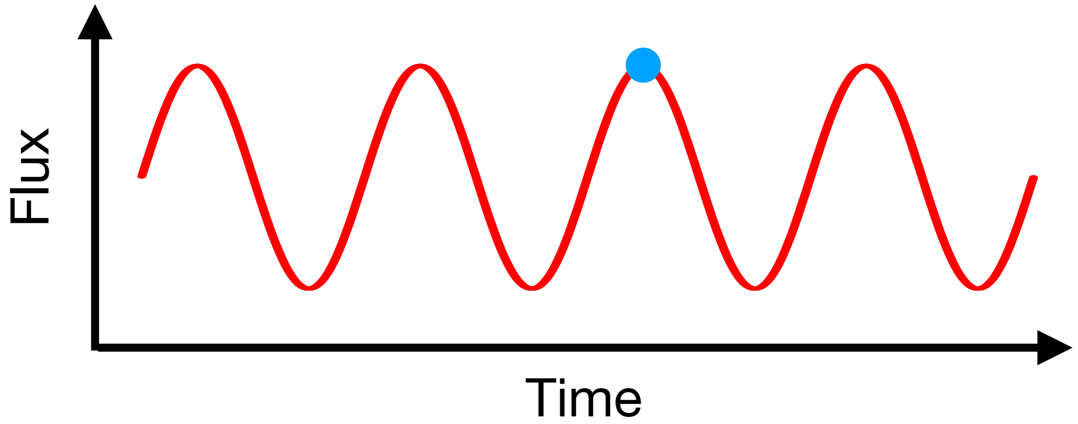
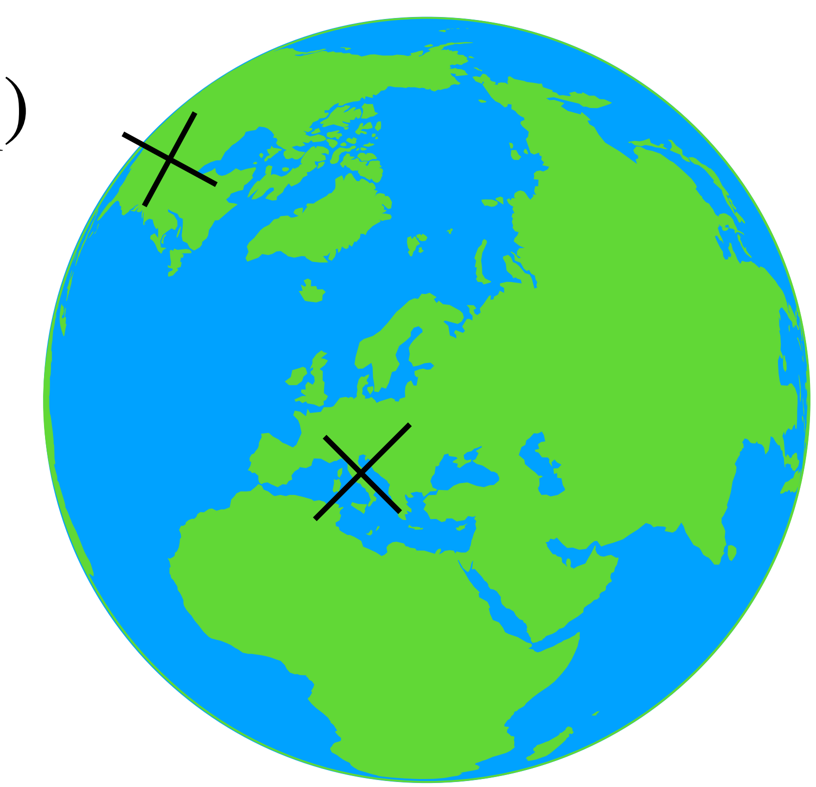


LIGO/Virgo

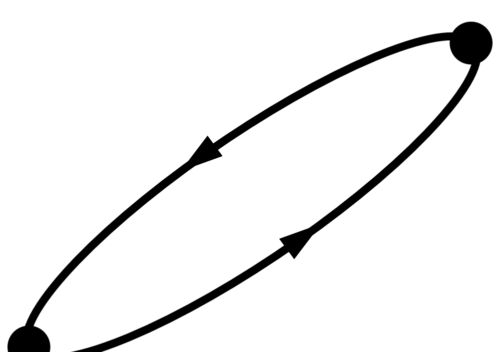
Different detectors see different signals because:

- Path length difference means GWs arrive at one detector slightly after another detector.
- Detectors have different orientations:

$$(s_{\text{arm}})^2 = g_{\mu\nu} dx^\mu dx^\nu \qquad dx^\mu = (0, \Delta x_{\text{arm}}, \Delta y_{\text{arm}}, \Delta z_{\text{arm}})$$

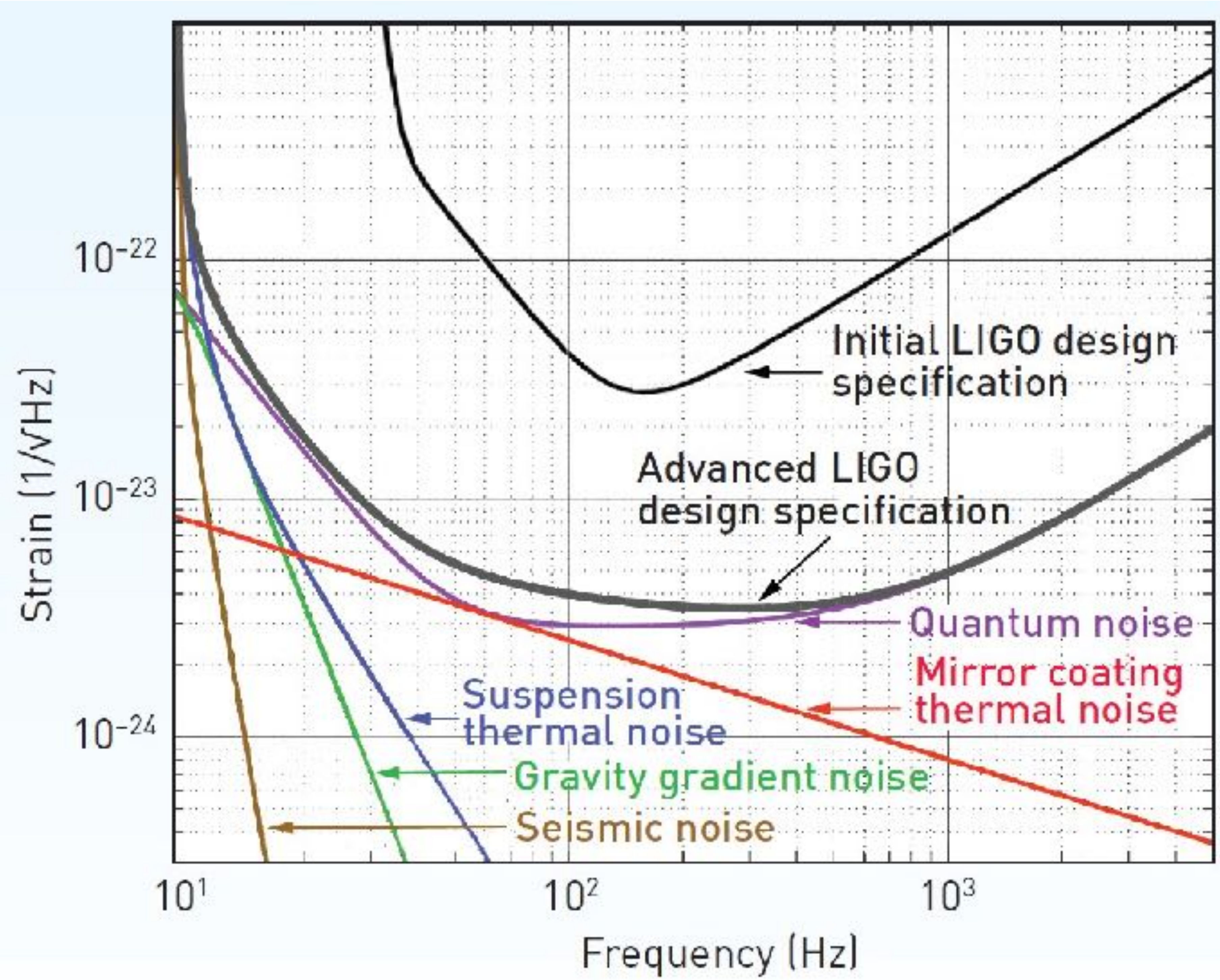


Both differences can be used for **verification** and **localisation**.



LIGO/Virgo

LIGO sensitivity is **amazing**, and needs to be to detect such a tiny signal!



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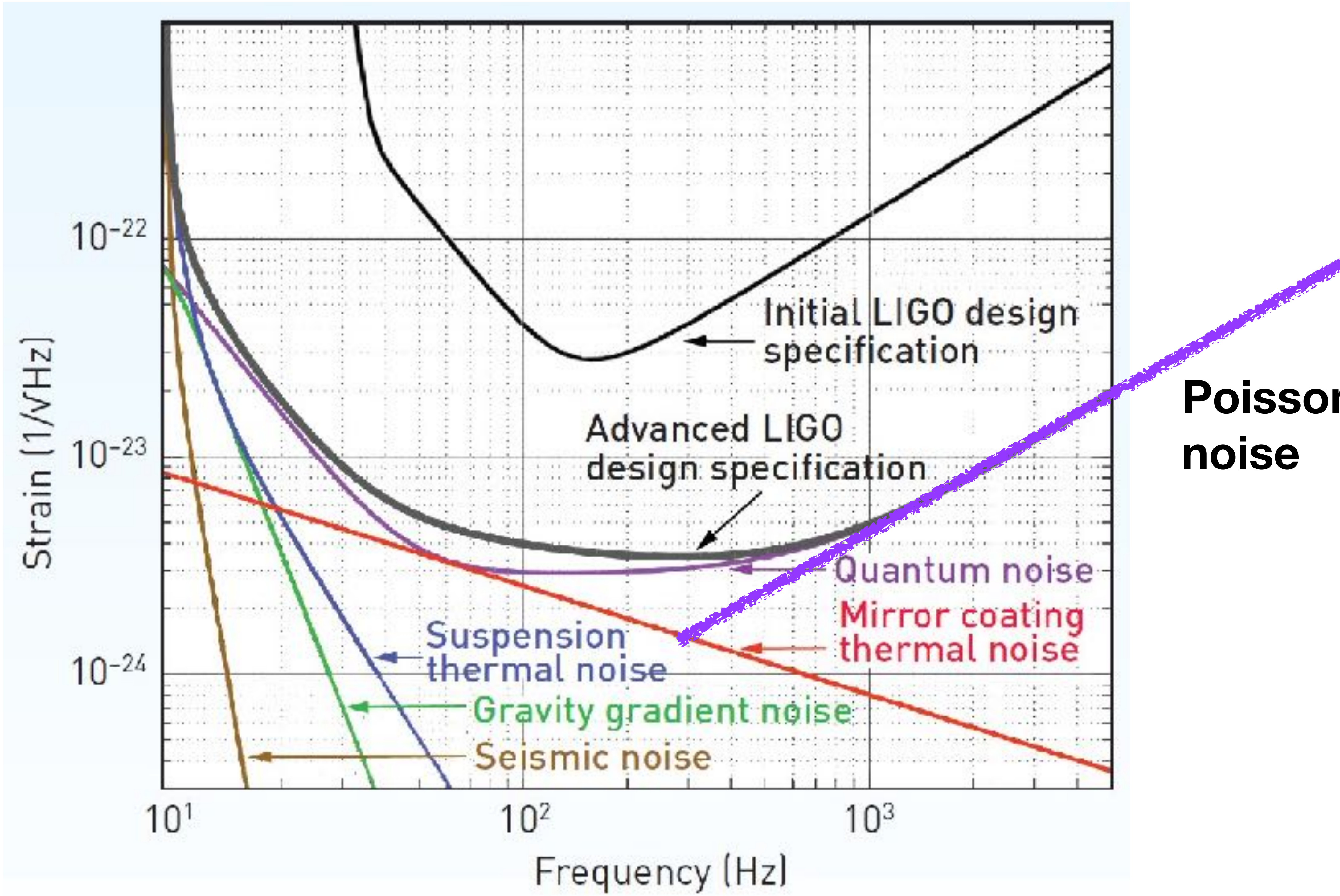
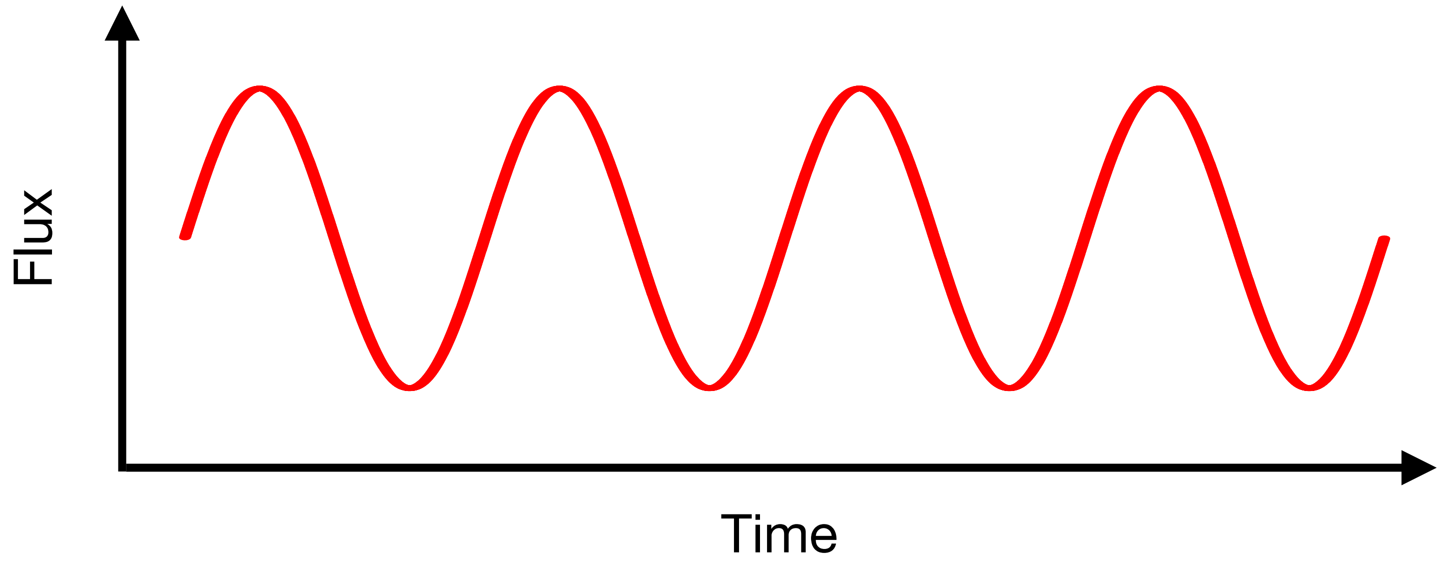


Poisson noise

LIGO/Virgo

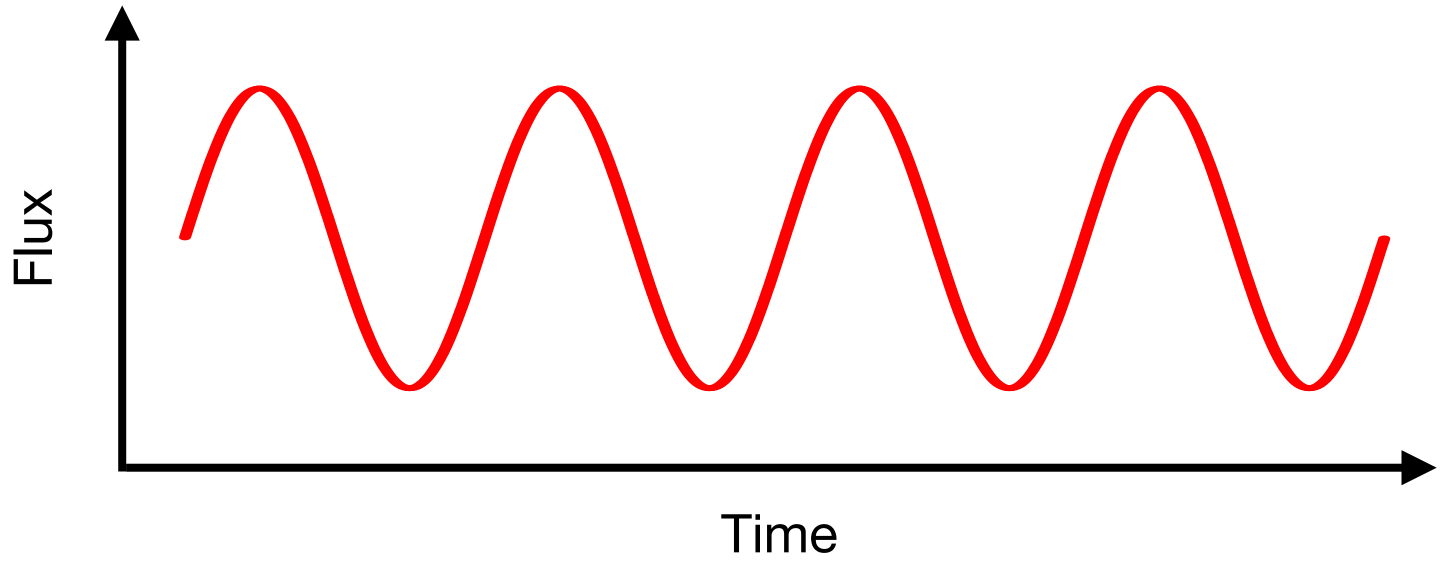
Poisson noise



- Need to accurately measure flux in time interval < 1/f

LIGO/Virgo

Poisson noise



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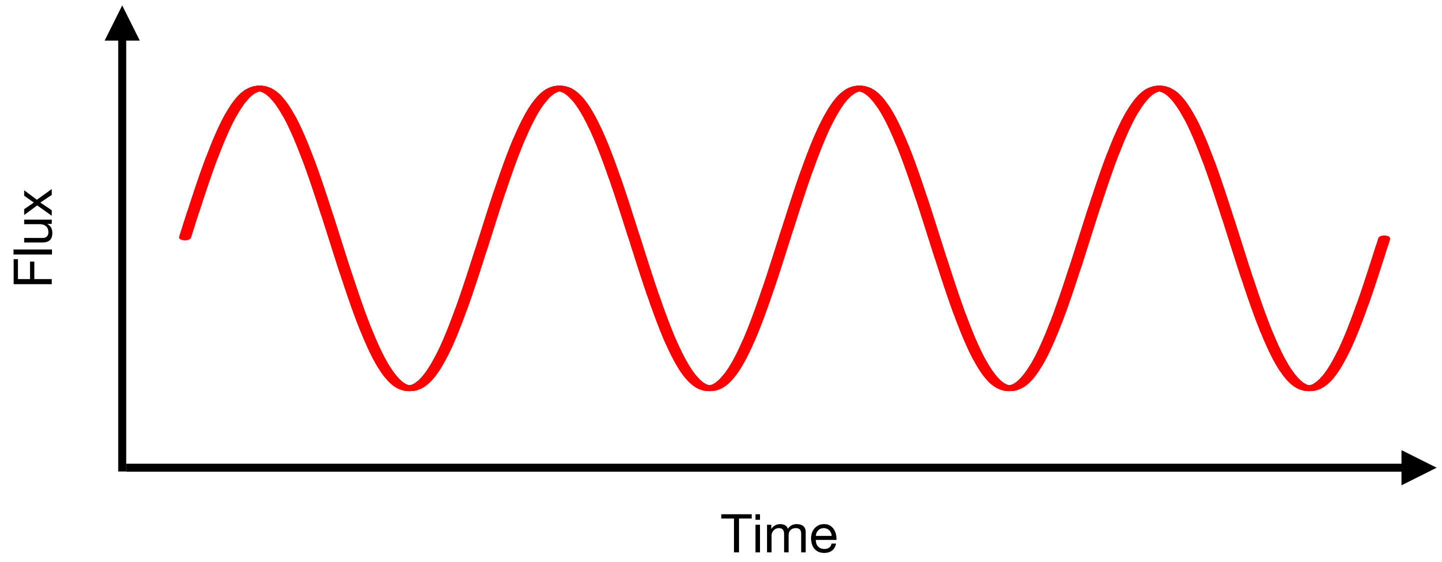
- No of photons in interval (1/f): $N = L(1/f)/(h\nu)$

LIGO/Virgo

Poisson noise



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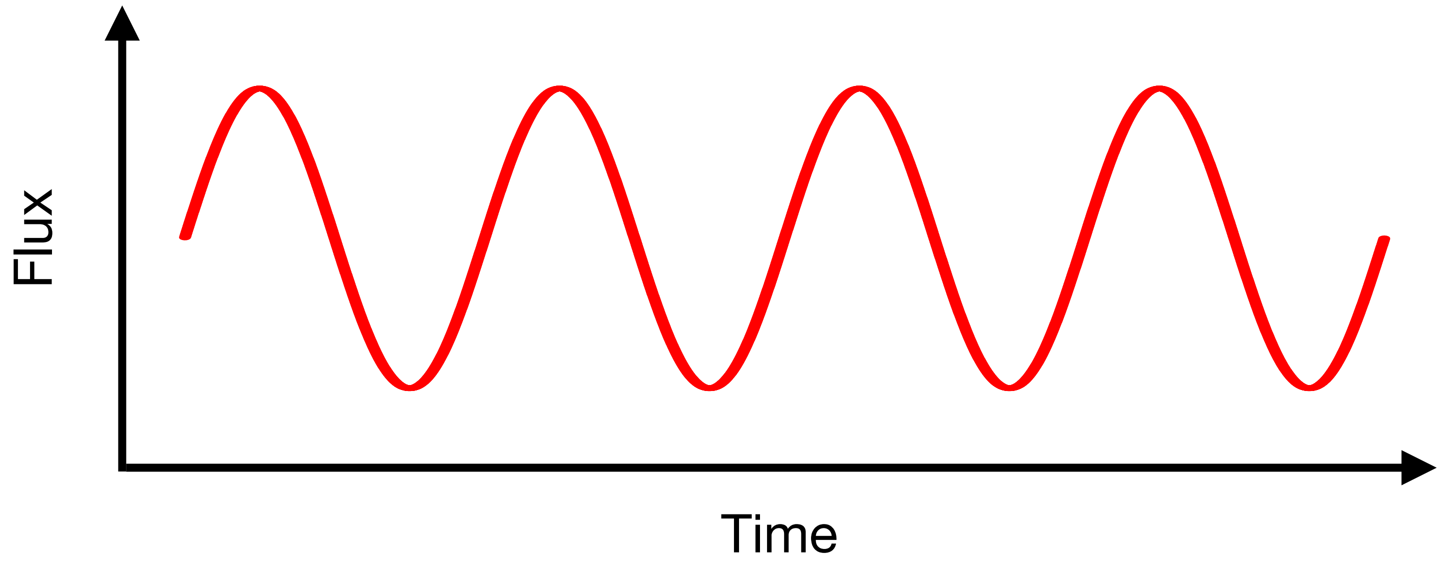
LIGO/Virgo

Poisson noise



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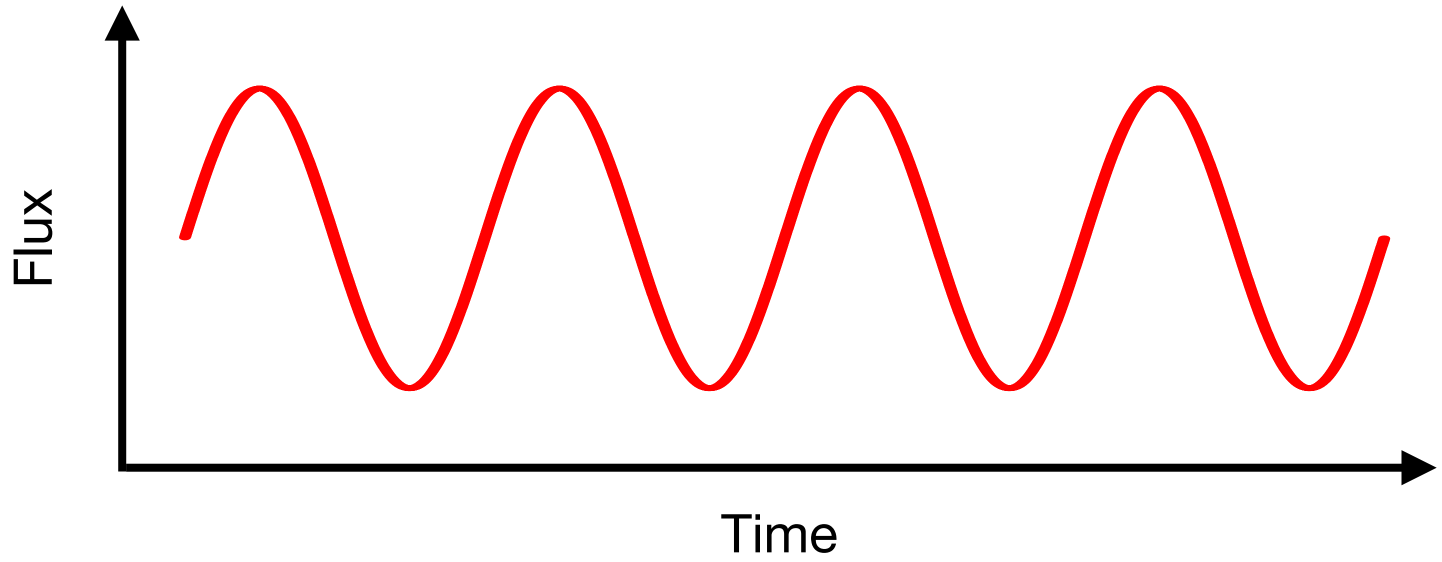
LIGO/Virgo

Poisson noise



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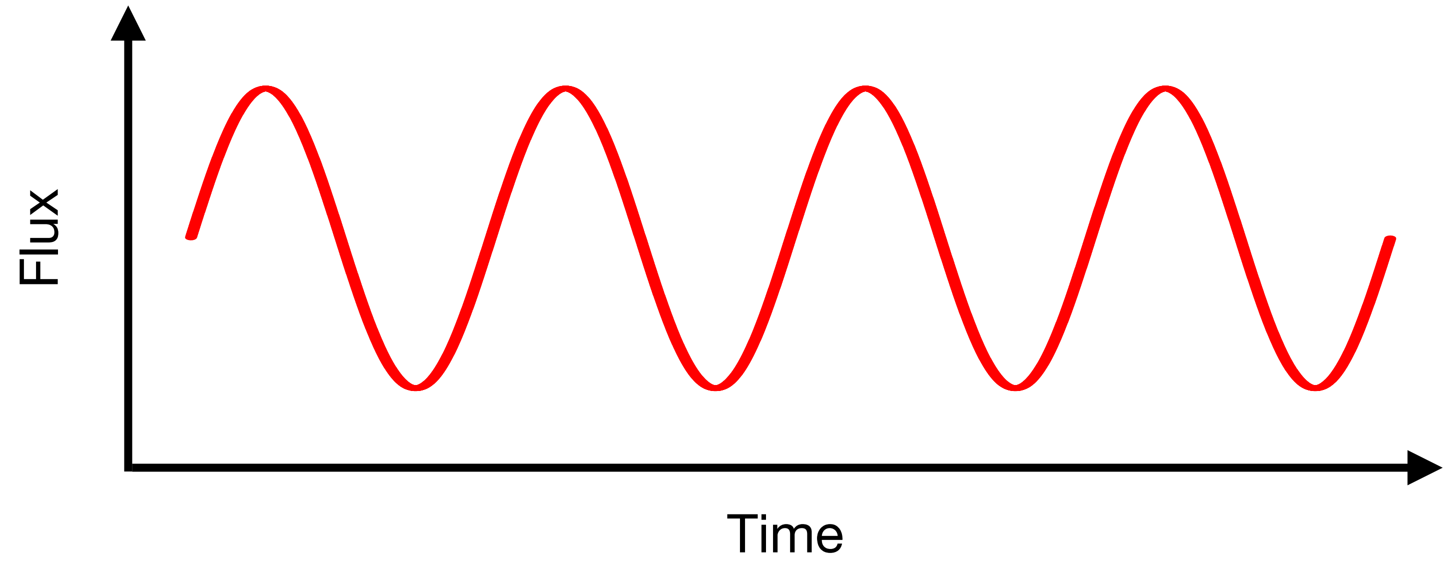
Could just crank up the laser power, L?

LIGO/Virgo

Poisson noise



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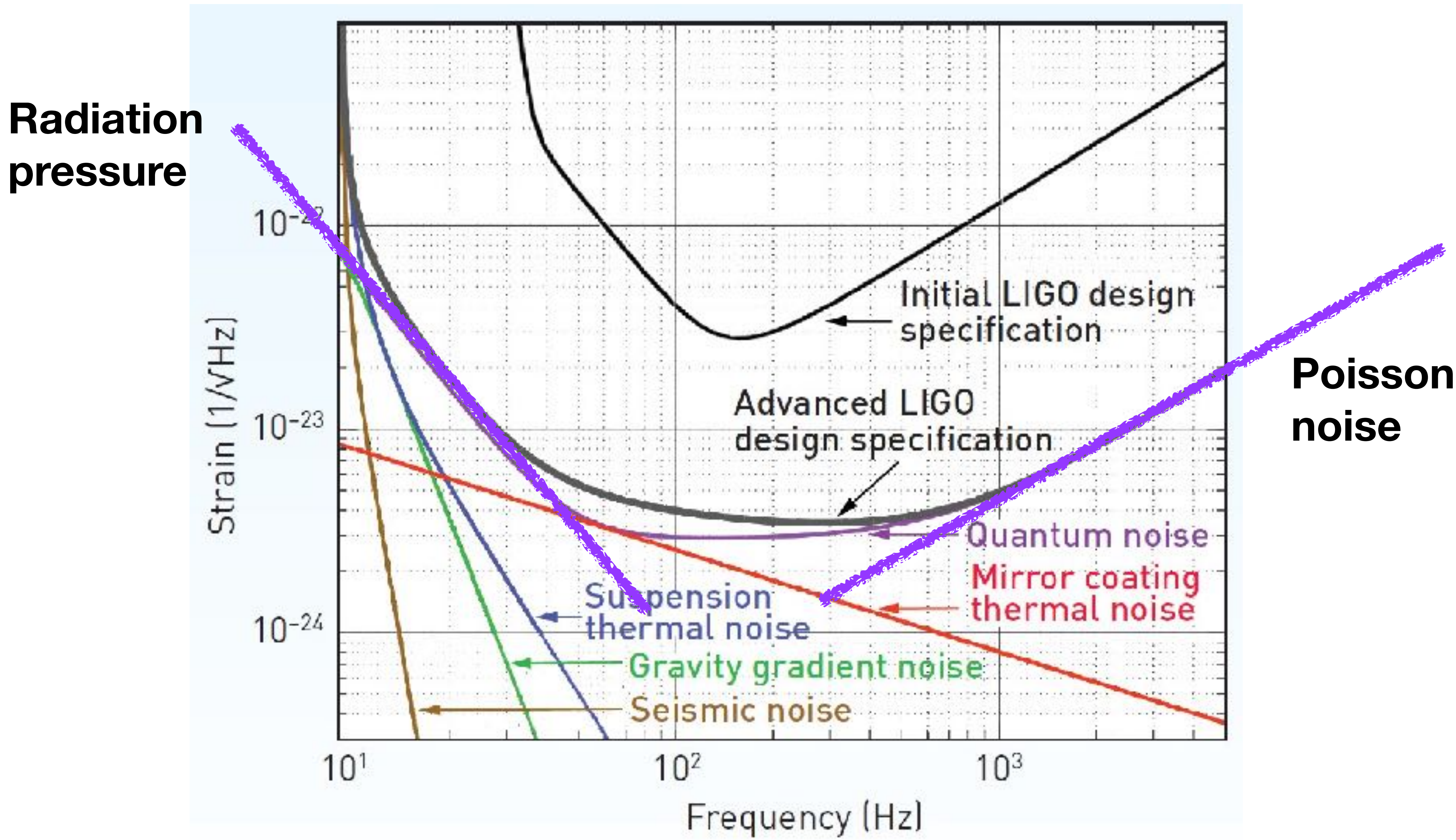
Could just crank up the laser power, L?

…but this increases radiation pressure, which wobbles the mirrors at lower frequencies!

LIGO/Virgo

LIGO sensitivity is **amazing**, and needs to be to detect such a tiny signal!



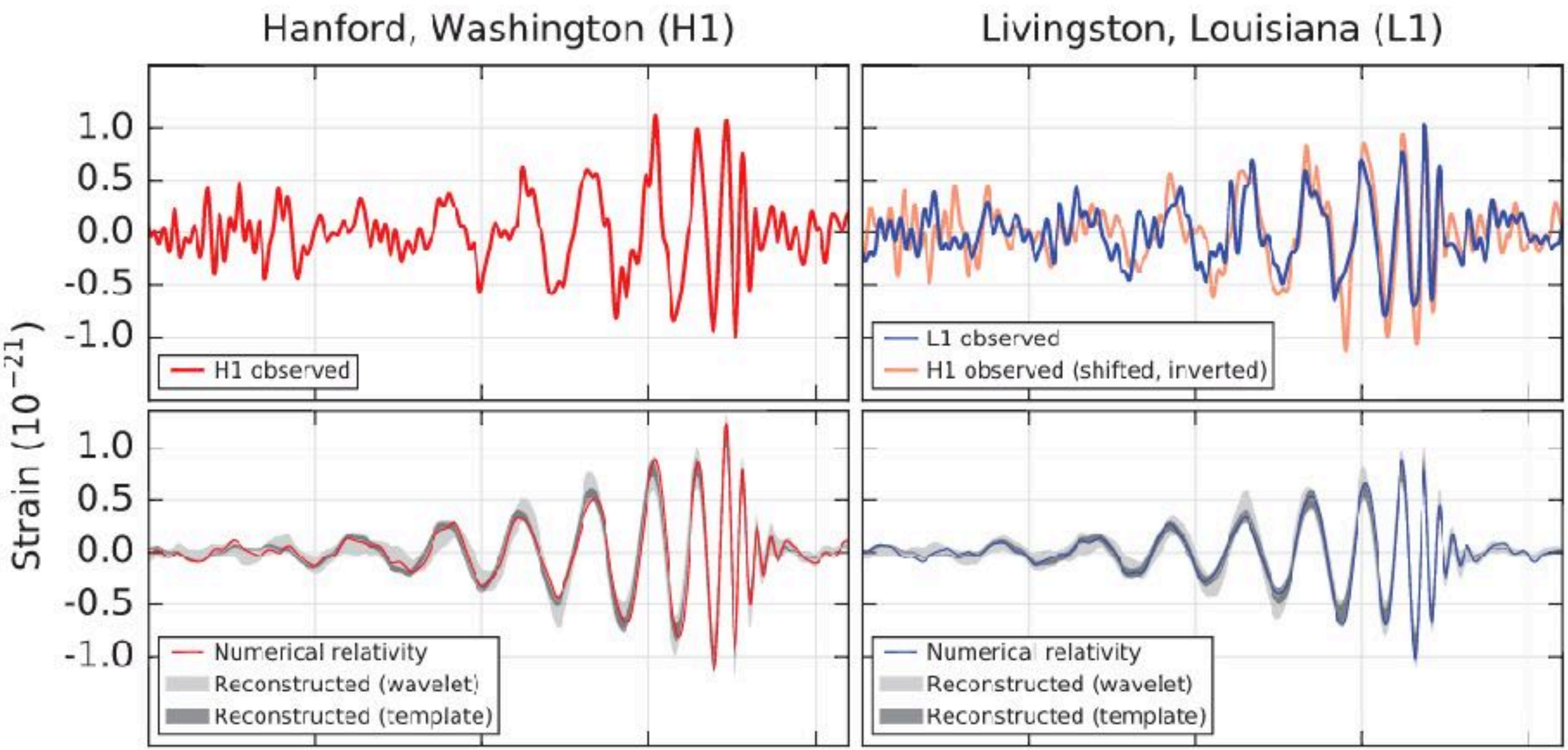
Laser power optimised to be most sensitive to GW f of merging BHs

LIGO/Virgo

The first GW event

LIGO/Virgo

The first GW event



14th September 2015

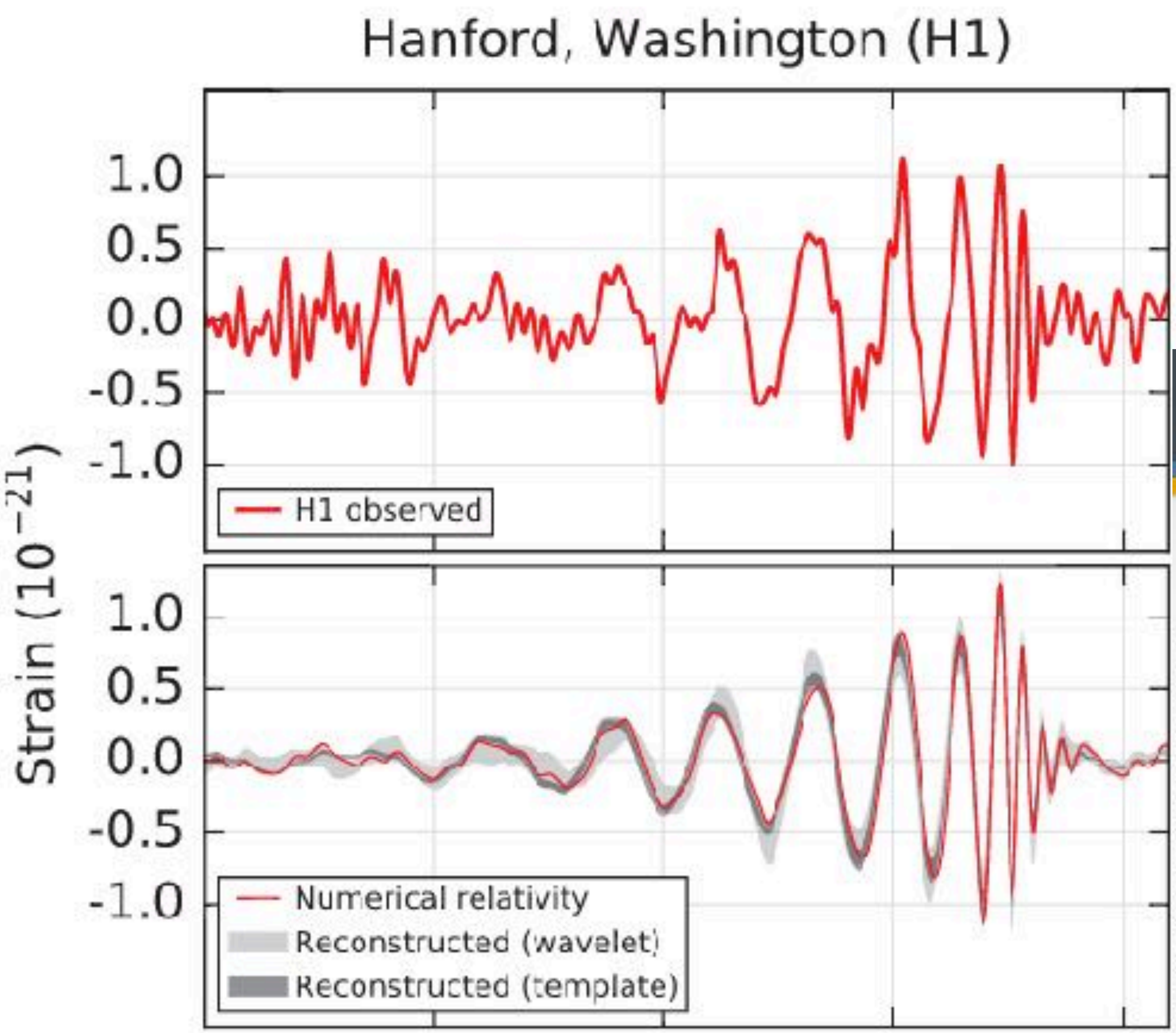
$$M_1 = 36^{+5}_{-4}\ M_\odot;\quad M_2 = 29^{+4}_{-4}\ M_\odot;\quad r = 410^{+160}_{-180}\ \mathrm{Mpc}$$

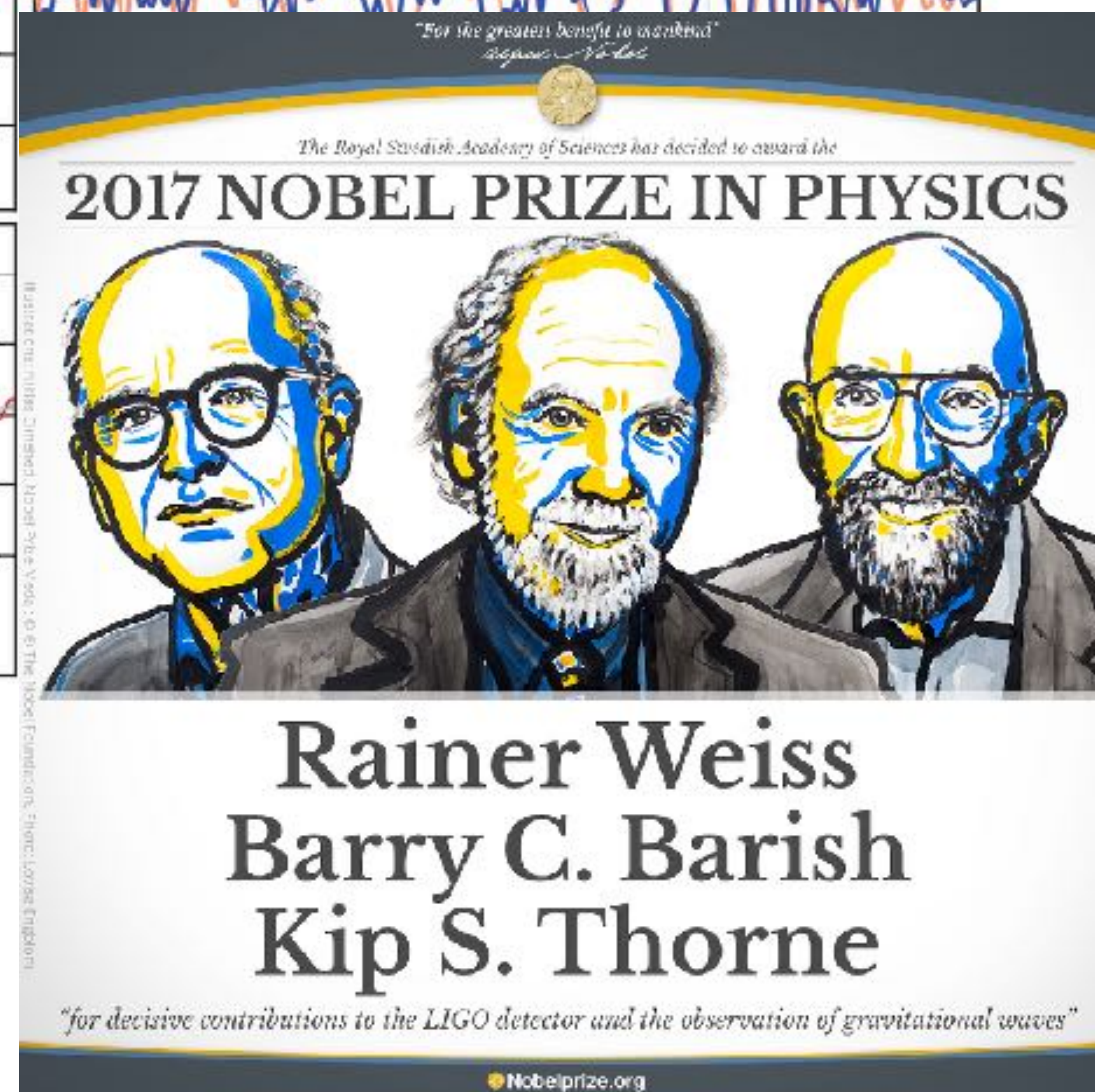
$$M_{\mathrm{product}} = 62^{+4}_{-4}\ M_\odot$$

…3 Msun of rest mass energy radiated away!

LIGO/Virgo

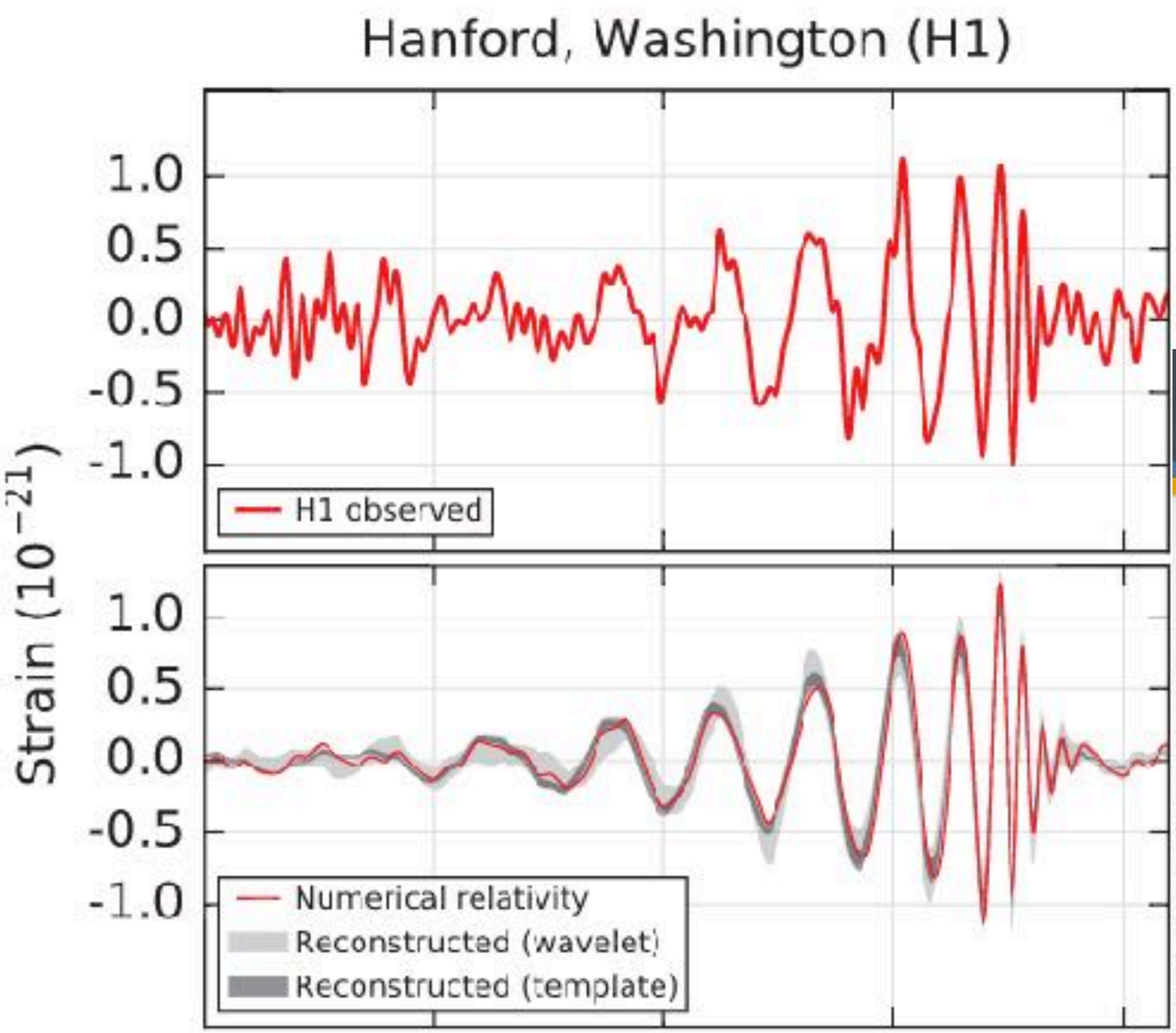
The first GW event

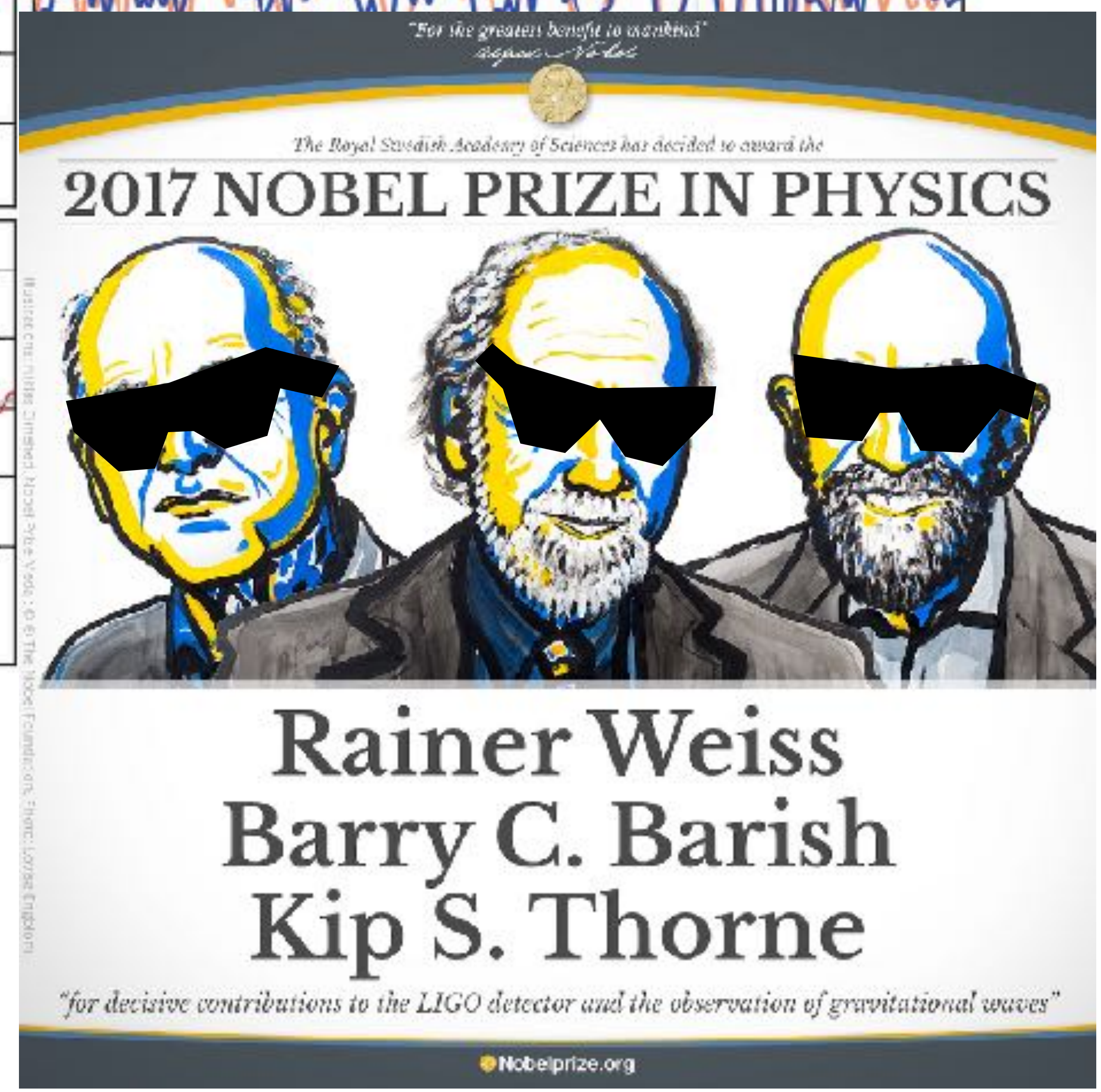




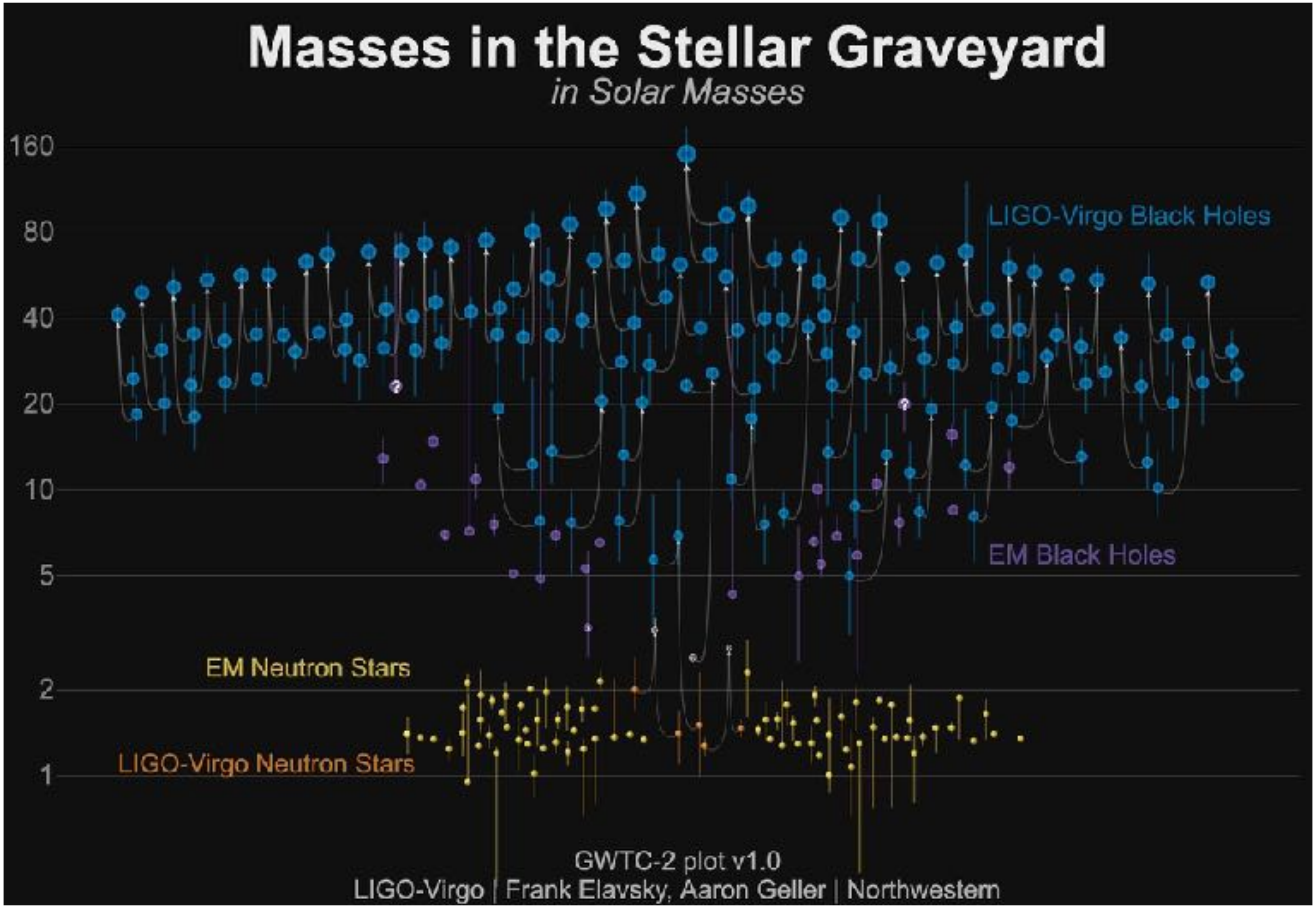
LIGO/Virgo

The first GW event





LIGO/Virgo Results after observing run 3



- Mainly BBHs, two BNSs and one of them had EM counterpart!
- GW BHs heavier than XRB BHs. Why? Different formation channels (LMXRBs will not become GW sources), BBH's progenitors formed a **long** time ago so lower metallicity.